Domain Theory

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Domain Theory

- advanced constructions on complete partial orders (cpo's)
- a meta-language for complete partial orders

Today's Topic

- advanced constructions on complete partial orders (cpo's)

Recap: Complete Partial Orders

Textbook, Chapter 5.4, Chapter 8.1

Complete Partial Orders

Recall: Partial Orders

A partial order is an ordered pair $(P, \sqsubseteq)$ such that P is a set and $\sqsubseteq$ is a binary relation $\sqsubseteq \subseteq P \times P$ satisfying the following conditions:

- (reflexibility) $\forall p \in P. p \sqsubseteq p$;
- (transitivity) $\forall p, q, r \in P. [(p \sqsubseteq q \ \& \ q \sqsubseteq r) \Rightarrow p \sqsubseteq r]$;
- (antisymmetry) $\forall p, q \in P. [(p \sqsubseteq q \ \& \ q \sqsubseteq p) \Rightarrow p = q]$.

Complete Partial Orders

Recall: Upper Bounds

- $(P, \sqsubseteq)$: a partial order
- X: a subset of P (i.e., that satisfies $X \subseteq P$)

$p \in P$ is an upper bound of X if $\forall q \in X. q \sqsubseteq p$.

Complete Partial Orders

Recall: Upper Bounds

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Recall: Least Upper Bounds

$p \in P$ is a least upper bound (in short, lub) of X if

- p is an upper bound of X, and
- for all upper bounds q of X, $p \sqsubseteq q$

Complete Partial Orders

Recall: Least Upper Bounds

$p \in P$ is a least upper bound (in short, lub) of X if

- p is an upper bound of X, and
- for all upper bounds q of X, $p \sqsubseteq q$

Recall: Notation

- The least upper bound of X (if exists) is denoted by $\bigsqcup X$.
- If $X = \{d_1, \ldots, d_n\}$, then $d_1 \sqcup \cdots \sqcup d_n := \bigsqcup X$.

Complete Partial Orders

Recall: ω-Chains

- $(P, \sqsubseteq)$: a partial order

An ω-chain in P is an infinite sequence $d_0, d_1, \ldots, d_n, \ldots$ in P such that $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$.

Complete Partial Orders

Recall: ω-Chains

- $(P, \sqsubseteq)$: a partial order

An ω-chain in P is an infinite sequence $d_0, d_1, \ldots, d_n, \ldots$ in P such that $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$.

Recall: Complete Partial Orders (CPOs)

$(P, \sqsubseteq)$ is a complete partial order (cpo) if for any ω-chain

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$$

in P, the least upper bound

$$\bigsqcup_{n \in \omega} d_n := \bigsqcup \{d_n \mid n \in \omega\} = \bigsqcup \{d_0, d_1, \ldots, d_n, \ldots\}$$

exists in P.

Complete Partial Orders

Recall: Least Elements

- $(P, \sqsubseteq)$: a partial order

$p \in P$ is a least element if $\forall q \in P. p \sqsubseteq q$.

Complete Partial Orders

Recall: Least Elements

- $(P, \sqsubseteq)$: a partial order

$p \in P$ is a least element if $\forall q \in P. p \sqsubseteq q$.

Recall: CPOs with Bottom

- $(P, \sqsubseteq)$: a cpo

$(P, \sqsubseteq)$ is a cpo with bottom if P has a (unique) least element $\bot_P$.

Complete Partial Orders

Recall: Set Inclusion

- A: a set
- $D := 2^A$
- $\sqsubseteq := \{(X, Y) \in D \times D \mid X \subseteq Y\}$

Complete Partial Orders

Recall: Partial Functions

- B, C: sets
- $D := \{F \mid F : B \rightharpoonup C\}$
- $\sqsubseteq := \{(F, G) \in D \times D \mid F \subseteq G\}$

Complete Partial Orders

Monotonic Functions

- $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$: partial orders

A function $f : D \to E$ is monotonic if

$$\forall d, d' \in D. \left[d \sqsubseteq_D d' \Rightarrow f(d) \sqsubseteq_E f(d') \right]$$

Continuous Functions

Definition

- $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$: cpo's

A function $f : D \to E$ is continuous if the followings hold:

- f is monotonic;
- for all ω-chains $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in D, we have that

$$\bigsqcup_{n \in \omega} f(d_n) = f\left(\bigsqcup_{n \in \omega} d_n\right)$$

Fixed Points

Definition

- $(D, \sqsubseteq_D)$: a partial order
- $f : D \to D$: a function

An element $d \in D$ is:

- a fixed point of f if $f(d) = d$;
- a prefixed point of f if $f(d) \sqsubseteq d$;

Complete Partial Orders

The Fixed-Point Theorem

- $(D, \sqsubseteq_D)$: a cpo with bottom $\bot_D$
- $f : D \to D$: a continuous function
- $\bot_D \sqsubseteq_D f(\bot_D) \sqsubseteq_D \cdots \sqsubseteq_D f^n(\bot_D) \sqsubseteq_D \cdots$
- $fix(f) := \bigsqcup_{n \in \omega} f^n(\bot_D)$

Complete Partial Orders

The Fixed-Point Theorem

- $(D, \sqsubseteq_D)$: a cpo with bottom $\bot_D$
- $f : D \to D$: a continuous function
- $\bot_D \sqsubseteq_D f(\bot_D) \sqsubseteq_D \cdots \sqsubseteq_D f^n(\bot_D) \sqsubseteq_D \cdots$
- $fix(f) := \bigsqcup_{n \in \omega} f^n(\bot_D)$

Then

- $fix(f)$ is a fixed point of f: $f(fix(f)) = fix(f)$
- $fix(f)$ is the least prefixed point of f: $f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d$
- $fix(f)$ is the least fixed point of f: $f(d) = d \Rightarrow fix(f) \sqsubseteq d$

Complete Partial Orders

Isomorphisms

- $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$: cpo's

A continuous function $f : D \to E$ is an isomorphism if

- f is a $1-1$ correspondence;
- $\forall d, d' \in D. [d \sqsubseteq_D d' \Leftrightarrow f(d) \sqsubseteq_E f(d')]$

Complete Partial Orders

Exercise

- D, E, F: cpo's (with their implicit ordering relations)
- $f : D \to E$ and $g : E \to F$: continuous functions

Then:

- the identity function $\mathrm{Id}_D : D \to D$ (such that $\mathrm{Id}_D(d) = d$ for all $d \in D$) is continuous;
- the function $g \circ f : D \to F$ is continuous.

Constructions on CPO's

Textbook, Chapter 8.3

Discrete CPO's

Textbook, Chapter 8.3.1

Discrete CPO's

Discrete CPO's

A discrete cpo is a partial order $(D, \sqsubseteq)$ such that $\sqsubseteq$ is the identity relation on D, i.e., $\sqsubseteq = \{(d, d) \mid d \in D\}$.

Discrete CPO's

Discrete CPO's

A discrete cpo is a partial order $(D, \sqsubseteq)$ such that $\sqsubseteq$ is the identity relation on D, i.e., $\sqsubseteq = \{(d, d) \mid d \in D\}$.

Exercise

Prove that the discrete cpo $(D, \sqsubseteq)$ above is indeed a cpo.

Discrete CPO's

Exercise

- D: a discrete cpo
- E: a cpo

Prove that any function $f : D \to E$ is continuous.

Product CPO's

Textbook, Chapter 8.3.2

Product CPO's

Cartesian Product $D_1 \times \cdots \times D_k$

- $D_1, \ldots, D_k$: cpo's
- $D_1 \times \cdots \times D_k$: the Cartesian product of $D_1, \ldots, D_k$,

$$(d_1, \ldots, d_k) \in D_1 \times \cdots \times D_k \text{ iff } d_i \in D_i \text{ for } i = 1, \ldots, k$$

Product CPO's

Cartesian Product $D_1 \times \cdots \times D_k$

- $D_1, \ldots, D_k$: cpo's
- $D_1 \times \cdots \times D_k$: the Cartesian product of $D_1, \ldots, D_k$,

$$(d_1, \ldots, d_k) \in D_1 \times \cdots \times D_k \text{ iff } d_i \in D_i \text{ for } i = 1, \ldots, k$$

Then the product cpo $(D, \sqsubseteq)$ is given by:

- $D := D_1 \times \cdots \times D_k$;
- $(d_1, \ldots, d_k) \sqsubseteq (d'_1, \ldots, d'_k)$ iff $d_i \sqsubseteq d'_i$ for $i = 1, \ldots, k$

Product CPO's

Exercise

For any ω-chain

$$(d_{1,0}, \ldots, d_{k,0}) \sqsubseteq (d_{1,1}, \ldots, d_{k,1}) \sqsubseteq \ \ldots \ \sqsubseteq (d_{1,n}, \ldots, d_{k,n}) \sqsubseteq \ldots$$

in $D_1 \times \cdots \times D_k$, we have

$$\bigsqcup_{n \in \omega} (d_{1,n}, \ldots, d_{k,n}) = (\bigsqcup_{n \in \omega} d_{1,n}, \ldots, \bigsqcup_{n \in \omega} d_{k,n}).$$

Product CPO's

Exercise

For any ω-chain

$$(d_{1,0}, \ldots, d_{k,0}) \sqsubseteq (d_{1,1}, \ldots, d_{k,1}) \sqsubseteq \ldots \sqsubseteq (d_{1,n}, \ldots, d_{k,n}) \sqsubseteq \ldots$$

in $D_1 \times \cdots \times D_k$, we have

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Proposition

$D_1 \times \cdots \times D_k$ is a cpo.

Product CPO's

The Projection Function

- $D_1, \ldots, D_k$: cpo's
- $D_1 \times \cdots \times D_k$: the Cartesian product of $D_1, \ldots, D_k$

Define the projection functions

$$\pi_i : D_1 \times \cdots \times D_k \to D_i \ (i = 1, \ldots, k)$$

by

$$\pi_i(d_1, \ldots, d_k) := d_i.$$

Product CPO's

The Projection Function

- $D_1, \ldots, D_k$: cpo's
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Exercise

Prove that each π_i is continuous.

Product CPO's

Tupling Function

- $E, D_1, \ldots, D_k$: cpo's
- $f_i : E \to D_i$ $(i = 1, \ldots, k)$: continuous functions

Define the tupling function

$$\langle f_1, \ldots, f_k \rangle : E \to D_1 \times \cdots \times D_k$$

by

$$\langle f_1, \ldots, f_k \rangle(e) := (f_1(e), \ldots, f_k(e))$$

Product CPO's

The Continuity of $\langle f_1, \ldots, f_k \rangle$

For any ω-chain

$$e_1 \sqsubseteq e_2 \sqsubseteq \cdots \sqsubseteq e_n \sqsubseteq \ldots$$

we have

$$\begin{aligned}
\langle f_1, \ldots, f_k \rangle (\bigsqcup_{n \in \omega} e_n) &= (f_1(\bigsqcup_{n \in \omega} e_n), \ldots, f_k(\bigsqcup_{n \in \omega} e_n)) \\
&= (\bigsqcup_{n \in \omega} f_1(e_n), \ldots, \bigsqcup_{n \in \omega} f_k(e_n)) \\
&= \bigsqcup_{n \in \omega} (f_1(e_n), \ldots, f_k(e_n)) \\
&= \bigsqcup_{n \in \omega} \langle f_1, \ldots, f_k \rangle (e_n)
\end{aligned}$$

Product CPO's

Exercise

- $D_1, \ldots, D_k, E_1, \ldots, E_k$: cpo's
- $f_i : D_i \to E_i$ $(i = 1, \ldots, k)$: continuous functions

Define the function

$$f_1 \times \cdots \times f_k : D_1 \times \cdots \times D_k \to E_1 \times \cdots \times E_k$$

by

$$f_1 \times \cdots \times f_k(d_1, \ldots, d_k) := (f_1(d_1), \ldots, f_k(d_k)).$$

Prove that $f_1 \times \cdots \times f_k$ is continuous.

Product CPO's

Lemma

- $E, D_1, \ldots, D_k$: cpo's
- $h : E \to D_1 \times \cdots \times D_k$: a function

Then h is continuous iff for $i = 1, \ldots, k$, the functions $\pi_i \circ h : E \to D_i$ are continuous.

Product CPO's

Lemma

- $E, D_1, \ldots, D_k$: cpo's
- $h : E \to D_1 \times \cdots \times D_k$: a function

Then h is continuous iff for $i = 1, \ldots, k$, the functions $\pi_i \circ h : E \to D_i$ are continuous.

Proof

"$\Rightarrow$": From compositionality of continuous functions.

"$\Leftarrow$": For all $e \in E$,

$$\begin{aligned} h(e) &= (\pi_1(h(e)), \ldots, \pi_k(h(e))) \\ &= (\pi_1 \circ h(e), \ldots, \pi_k \circ h(e)) \\ &= \langle \pi_1 \circ h, \ldots, \pi_k \circ h \rangle (e) \end{aligned}$$

Product CPO's

Lemma

- $E, D_1, \ldots, D_k$: cpo's
- $f : D_1 \times \cdots \times D_k \to E$: a function

Then f is continuous iff for all $1 \leq i \leq k$ and all elements

$$d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k$$

we have that the function $h_i : D_i \to E$ defined as

$$d \mapsto f(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_k)$$

is continuous.

Product CPO's

Proposition

- E: a cpo
- $e_{n,m}$ $(n \in \omega, m \in \omega)$: elements of the cpo E
- $e_{n,m} \sqsubseteq e_{n',m'}$ whenever $n \leq n'$ and $m \leq m'$

Product CPO's

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- $e_{n,m} \sqsubseteq e_{n',m'}$ whenever $n \leq n'$ and $m \leq m'$

Then we have that

- The set $\{e_{n,m} \mid n, m \in \omega\}$ has a (unique) least upper bound.
- $$\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n \in \omega} (\bigsqcup_{m \in \omega} e_{n,m}) = \bigsqcup_{m \in \omega} (\bigsqcup_{n \in \omega} e_{n,m}) = \bigsqcup_{n \in \omega} e_{n,n}$$

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Proof

- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n,n \in \omega} e_{n,n}$;

Product CPO's

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Proof

- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n,n \in \omega} e_{n,n}$;
- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n \in \omega} (\bigsqcup_{m \in \omega} e_{n,m})$;

Product CPO's

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Proof

- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n,n \in \omega} e_{n,n}$;
- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{n \in \omega} (\bigsqcup_{m \in \omega} e_{n,m})$;
- $\bigsqcup_{n,m \in \omega} e_{n,m} = \bigsqcup_{m \in \omega} (\bigsqcup_{n \in \omega} e_{n,m})$;

Product CPO's

Lemma

- $E, D_1, \ldots, D_k$: cpo's
- $f : D_1 \times \cdots \times D_k \to E$: a function

Then f is continuous iff for all $1 \leq i \leq k$ and all elements

$$d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k$$

we have that the function $h_i : D_i \to E$ defined as

$$d \mapsto f(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_k)$$

is continuous.

Product CPO's

Proof

"$\Rightarrow$": Straightforward.

Product CPO's

Proof

"$\Rightarrow$": Straightforward.

"$\Leftarrow$": The monotonicity is straightforward. For the rest of the proof, we take $k = 2$. Consider any ω-chain

$$(x_0, y_0) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \ldots$$

in $D_1 \times D_2$. Then

$$\begin{aligned} f(\bigsqcup_n (x_n, y_n)) &= f(\bigsqcup_n x_n, \bigsqcup_m y_m) \\ &= \bigsqcup_n f(x_n, \bigsqcup_m y_m) \\ &= \bigsqcup_n \bigsqcup_m f(x_n, y_m) \\ &= \bigsqcup_n f(x_n, y_n) \end{aligned}$$

Function Space

Textbook, Chapter 8.3.3

Function Space

Definitions

- D, E: cpo's

Function Space

Definitions

- D, E: cpo's
- $[D \to E] := \{f \mid f : D \to E \text{ is continuous.}\}$.

Function Space

Definitions

- D, E: cpo's
- $[D \to E] := \{f \mid f : D \to E \text{ is continuous.}\}$.
- For $f, g \in [D \to E]$, $f \sqsubseteq g$ iff $\forall d \in D. f(d) \sqsubseteq g(d)$.

Function Space

Theorem

$([D \to E], \sqsubseteq)$ is a complete partial order.

Proof

Consider any ω-chain

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \ldots$$

in $([D \to E], \sqsubseteq)$. Then the least upper bound $\bigsqcup_{n\in\omega} f_n$ is given by:

$$(\bigsqcup_{n\in\omega} f_n)(d) := \bigsqcup_{n\in\omega}(f_n(d)) \text{ for all } d \in D.$$

We still need to prove that $\bigsqcup_{n\in\omega} f_n \in [D \to E]$.

Function Space

Theorem

$([D \to E], \sqsubseteq)$ is a complete partial order.

Proof (Continued)

We still need to prove that $\bigsqcup_{n\in\omega} f_n \in [D \to E]$. For any ω-chain

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_m \sqsubseteq \ldots$$

in D, we have that

$$\begin{aligned}
(\bigsqcup_{n\in\omega} f_n)(\bigsqcup_{m\in\omega} d_m) &= \bigsqcup_{n\in\omega} f_n(\bigsqcup_{m\in\omega} d_m) \\
&= \bigsqcup_{n\in\omega} \bigsqcup_{m\in\omega} (f_n(d_m)) \\
&= \bigsqcup_{m\in\omega} \bigsqcup_{n\in\omega} (f_n(d_m)) \\
&= \bigsqcup_{m\in\omega} (\bigsqcup_{n\in\omega} f_n)(d_m).
\end{aligned}$$

Function Space

Bottom Element

- D, E: cpo's

Function Space

Bottom Element

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- $[D \to E] := \{f \mid f : D \to E \text{ is continuous.}\}$.

Function Space

Bottom Element

- D, E: cpo's
- $[D \to E] := \{f \mid f : D \to E \text{ is continuous.}\}$.
- For $f, g \in [D \to E]$, $f \sqsubseteq g$ iff $\forall d \in D. f(d) \sqsubseteq g(d)$.

If E has a bottom element $\bot_E$, then $[D \to E]$ also has a bottom element given by:

$$\bot_{[D \to E]}(d) := \bot_E \text{ for all } d \in D.$$

Function Space

Application

- D, E: cpo's

Define $apply : ([D \to E] \times D) \to E$ by:

$$apply(f, d) := f(d) \text{ for all } f \in [D \to E], d \in D.$$

Function Space

Application

- D, E: cpo's

Define $apply : ([D \to E] \times D) \to E$ by:

$$apply(f, d) := f(d) \text{ for all } f \in [D \to E], d \in D.$$

Theorem

The function $apply$ is continuous.

Function Space

Theorem

The function *apply* is continuous.

Proof

apply is continuous in its first argument:

- monotonicity;
- consider any ω-chain $f_0 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \ldots$ in $[D \to E]$;
- $apply(\bigsqcup_n f_n, d) = (\bigsqcup_n f_n)(d) = \bigsqcup_n (f_n(d)) = \bigsqcup_n apply(f_n, d)$.

Function Space

Theorem

The function *apply* is continuous.

Proof

apply is continuous in its first argument:

- monotonicity;
- consider any ω-chain $f_0 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \ldots$ in $[D \to E]$;
- $apply(\bigsqcup_n f_n, d) = (\bigsqcup_n f_n)(d) = \bigsqcup_n (f_n(d)) = \bigsqcup_n apply(f_n, d)$.

apply is continuous in its second argument:

- monotonicity;
- consider any $d_0 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in D;
- $apply(f_n, \bigsqcup_n d_n) = f(\bigsqcup_n d_n) = \bigsqcup_n (f(d_n)) = \bigsqcup_n apply(f, d_n)$.

Function Space

λ-Notation

- X, Y: sets
- $f : X \to Y$: a function
- e: an expression representing f (e.g., $e = x + 1$ and $f(x) = x + 1$)

Then we denote also by $\lambda x \in X.e$ the function f.

Examples

- $\lambda x \in \omega.(x + 1)$: the function $f(x) = x + 1$
- $\lambda x \in \mathbb{R}. \sin x$: the function $f(x) = \sin x$

Function Space

Currying

- D, E, F: cpo's
- $g : F \times D \to E$: a continuous function

Define the function $curry(g) : F \to [D \to E]$ by:

$$curry(g) := \lambda v \in F.(\lambda d \in D.g(v, d))$$

Function Space

Currying

- D, E, F: cpo's
- $g : F \times D \to E$: a continuous function

Define the function $curry(g) : F \to [D \to E]$ by:

$$curry(g) := \lambda v \in F.(\lambda d \in D.g(v, d))$$

Theorem

- For all $v \in F$, $curry(g)(v) \in [D \to E]$.
- $curry(g)$ is continuous.

Function Space

Theorem

- For all $v \in F$, $curry(g)(v) \in [D \to E]$.

Function Space

Theorem

- For all $v \in F$, $curry(g)(v) \in [D \to E]$.

Proof

- $curry(g)(v) = \lambda d \in D.g(v, d)$;
- g is continuous in its second argument.

Function Space

Theorem

- $curry(g)$ is continuous.

Function Space

Theorem

- $curry(g)$ is continuous.

Proof

- monotonicity;

Function Space

Theorem

- $curry(g)$ is continuous.

Proof

- monotonicity;
- Consider any ω-chain $v_0 \sqsubseteq v_1 \sqsubseteq \cdots \sqsubseteq v_n \sqsubseteq \ldots$ in F. Then for all $d \in D$,

$$\begin{aligned}
(curry(g)(\bigsqcup_n v_n))(d) &= g(\bigsqcup_n v_n, d) \\
&= \bigsqcup_n g(v_n, d) \\
&= \bigsqcup_n((curry(g)(v_n))(d)) \\
&= (\bigsqcup_n(curry(g)(v_n)))(d)
\end{aligned}$$

Function Space

Theorem

- $curry(g)$ is continuous.

Proof

- monotonicity;
- Consider any ω-chain $v_0 \sqsubseteq v_1 \sqsubseteq \cdots \sqsubseteq v_n \sqsubseteq \ldots$ in F. Then for all $d \in D$,

$$\begin{aligned}
(curry(g)(\bigsqcup_n v_n))(d) &= g(\bigsqcup_n v_n, d) \\
&= \bigsqcup_n g(v_n, d) \\
&= \bigsqcup_n((curry(g)(v_n))(d)) \\
&= (\bigsqcup_n(curry(g)(v_n)))(d)
\end{aligned}$$

- $curry(g)(\bigsqcup_n v_n) = \bigsqcup_n(curry(g)(v_n))$

Lifting

Textbook, Chapter 8.3.4

Lifting

Definition

- D: a cpo
- $\perp$: a fresh bottom element

Lifting

Definition

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- $\bot$: a fresh bottom element
- $\lfloor - \rfloor$: a copy function on D such that
  - for all $d, d' \in D$, $d = d' \Leftrightarrow \lfloor d \rfloor = \lfloor d' \rfloor$;
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- $D_\bot := \{\lfloor d \rfloor \mid d \in D\} \cup \{\bot\}$;
- for all $d'_0, d'_1 \in D_\bot$, $d'_0 \sqsubseteq d'_1$ iff
  - either $d'_0 = \bot$,
  - or $d'_0 = \lfloor d_0 \rfloor, d'_1 = \lfloor d_1 \rfloor$ and $d_0 \sqsubseteq_D d_1$.

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- for all $d_0', d_1' \in D_\perp$, $d_0' \sqsubseteq d_1'$ iff
  - either $d_0' = \perp$,
  - or $d_0' = \lfloor d_0 \rfloor, d_1' = \lfloor d_1 \rfloor$ and $d_0 \sqsubseteq_D d_1$.

Exercise

- $D_\perp$ is a cpo with bottom.
- $\lfloor - \rfloor : D \to D_\perp$ is continuous.

Lifting

The Operator $(-)^*$

- D: a cpo
- $\bot$: a fresh bottom element
- E: a cpo with the bottom element $\bot_E$
- $f : D \to E$: a continuous function

Define $f^* : D_\bot \to E$ by:

$$f^*(d') := \begin{cases} f(d) & \text{if } d' = \lfloor d \rfloor \text{ for some } d \in D \\ \bot_E & \text{otherwise (i.e. } d' = \bot) \end{cases}$$

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Then (exercise) f^* is continuous.

Lifting

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Consider any $d' \in D_\perp$:

- if $d' = \perp$, then $(\bigsqcup_{n\in\omega} f_n)^*(d') = (\bigsqcup_{n\in\omega}(f_n)^*)(d') = \perp_E$;
- if $d' = \lfloor d \rfloor$, then

$$\begin{aligned}(\bigsqcup_{n\in\omega} f_n)^*(d') &= (\bigsqcup_{n\in\omega} f_n)(d)\\ &= \bigsqcup_{n\in\omega}(f_n(d))\\ &= \bigsqcup_{n\in\omega}((f_n)^*(d'))\\ &= (\bigsqcup_{n\in\omega}(f_n)^*)(d')\end{aligned}$$

Lifting

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Thus $(\bigsqcup_{n\in\omega} f_n)^* = \bigsqcup_{n\in\omega}(f_n)^*$.

Lifting

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Define

$$let\ x \Leftarrow d'.e := (\lambda x \in D.e)^*(d') \text{ for } d' \in D_\perp.$$

Lifting

Abbreviation

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- $let\ x_1 \Leftarrow c_1, \cdots, x_k \Leftarrow c_k.\ e$

Lifting

Truth Values

- $\mathbf{T} = \{\mathbf{true}, \mathbf{false}\}$;
- $\vee : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$: the or-function (from the truth table);

Define $\vee_\perp : \mathbf{T}_\perp \times \mathbf{T}_\perp \to \mathbf{T}_\perp$ by:

$$x_1 \vee_\perp x_2 := \mathit{let}\ t_1 \Leftarrow x_1, t_2 \Leftarrow x_2.\lfloor t_1 \vee t_2 \rfloor \quad .$$

Lifting

Arithmetic Operations

$$x_1 +_\perp x_2 := \mathit{let}\ n_1 \Leftarrow x_1, n_2 \Leftarrow x_2 . \lfloor n_1 + n_2 \rfloor \quad .$$

Sums (Disjoint Unions)

Textbook, Chapter 8.3.5

Sums (Disjoint Unions)

Definition

- $D_1, \ldots, D_k$: cpo's
- $in_1, \ldots, in_k$: 1−1 injection functions that make disjoint copies

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Define $D_1 + \cdots + D_k$ to be the cpo as follows:

- the underlying set is the disjoint union

$$\{in_1(d_1) \mid d_1 \in D_1\} \cup \cdots \cup \{in_k(d_k) \mid d_k \in D_k\};$$

- $d_1 \sqsubseteq d_2$ iff

$$\exists 1 \leq i \leq k. \exists d_1', d_2'. (d_1 = in_i(d_1') \ \& \ d_2 = in_i(d_2') \ \& \ d_1' \sqsubseteq_{D_i} d_2')$$

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Exercise

- $D_1 + \cdots + D_k$ is a cpo.
- each $in_i : D_i \to D_1 + \cdots + D_k$ is continuous.

Sums (Disjoint Unions)

Combination of Continuous Functions

- $E, D_1, \ldots, D_k$: cpo's
- $f_i : D_i \to E$ $(i = 1, \ldots, k)$: continuous functions

Define $[f_1, \ldots, f_k] : D_1 + \cdots + D_k \to E$ by

$$[f_1, \ldots, f_k](in_i(d_i)) := f_i(d_i) \text{ for all } i \text{ and } d_i \in D_i.$$

Sums (Disjoint Unions)

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$$[f_1, \ldots, f_k](in_i(d_i)) := f_i(d_i) \text{ for all } i \text{ and } d_i \in D_i.$$

Exercise

- $[f_1, \ldots, f_k]$ is a continuous function.
- The map $(f_1, \ldots, f_k) \mapsto [f_1, \ldots, f_k]$ is continuous.

Sums (Disjoint Unions)

Conditional Branches

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- $\lambda x_1.e_1 : \{\mathbf{true}\} \to E$
- $\lambda x_2.e_2 : \{\mathbf{false}\} \to E$
- $cond(t, e_1, e_2) := [\lambda x_1.e_1, \lambda x_2.e_2](t)$
- $$cond(t, e_1, e_2) = \begin{cases} e_1 & \text{if } t = \mathbf{true} \\ e_2 & \text{if } t = \mathbf{false} \end{cases}$$

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-

$$(b \to e_1 \mid e_2) := \mathit{let}\ t \Leftarrow b.\mathit{cond}(t, e_1, e_2) = \begin{cases} e_1 & \text{if } b = \lfloor \mathbf{true} \rfloor \\ e_2 & \text{if } b = \lfloor \mathbf{false} \rfloor \\ \bot_E & \text{if } b = \bot \end{cases}$$

Sums (Disjoint Unions)

Case Construction

- $E, D_1, \ldots, D_k$: cpo's
- d: an element in $D_1 + \cdots + D_k$
- $\lambda x_i.e_i : D_i \to E$ $(i = 1, \ldots, k)$: continuous functions

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-

$$\textbf{case } d \textbf{ of } in_1(x_1).e_1 \mid$$
$$\vdots$$
$$in_k(x_k).e_k$$

Sums (Disjoint Unions)

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- $\lambda x_i.e_i : D_i \to E$ $(i = 1, \ldots, k)$: continuous functions
- $$\textbf{case } d \textbf{ of } in_1(x_1).e_1 \mid \vdots \mid in_k(x_k).e_k$$
- $[\lambda x_1.e_1, \ldots, \lambda x_k.e_k](d)$

Summary

Advanced Constructions for CPO's

- discrete cpo's
- product cpo's
- function space
- lifting
- sums (disjoint unions)

Questions?