Languages with Higher Types

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May 28th

Today's Topic

- eager denotational semantics
- lazy denotational semantics

Eager Denotational Semantics

Textbook, Chapter 11.3

Eager Denotational Semantics

An Overview

- terms as functions from environments to values
- cpo's and continuous functions as mathematical backbone

Eager Denotational Semantics

Values

- τ: a type

The cpo $V^{\mathfrak{e}}_{\tau}$ of values associated with the type τ is recursively defined as follows:

- $V^{\mathfrak{e}}_{\mathbf{int}} := \mathbb{Z}$ (discrete cpo);
- $V^{\mathfrak{e}}_{\tau_1 * \tau_2} := V^{\mathfrak{e}}_{\tau_1} \times V^{\mathfrak{e}}_{\tau_2}$ (product cpo);
- $V^{\mathfrak{e}}_{\tau_1 \to \tau_2} := [V^{\mathfrak{e}}_{\tau_1} \to (V^{\mathfrak{e}}_{\tau_2})_{\perp}]$ (function space);

Eager Denotational Semantics

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The cpo V^{e}_{τ} of values associated with the type τ is recursively defined as follows:

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- $V^{\mathrm{e}}_{\tau_1 * \tau_2} := V^{\mathrm{e}}_{\tau_1} \times V^{\mathrm{e}}_{\tau_2}$ (product cpo);
- $V^{\mathrm{e}}_{\tau_1 \to \tau_2} := [V^{\mathrm{e}}_{\tau_1} \to (V^{\mathrm{e}}_{\tau_2})_{\perp}]$ (function space);

Question

Why do we incorporate $\perp$ in the last definition?

Eager Denotational Semantics

Environments

- **Var**: the set of variables

An *environment* ρ is a function

$$\rho : \mathbf{Var} \to \bigcup \{ V_\tau^e \mid \tau \text{ a type} \}$$

such that

$$\forall x \in \mathbf{Var}.(x : \tau \Rightarrow \rho(x) \in V_\tau^e) \ .$$

Eager Denotational Semantics

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such that

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We denote by $\mathbf{Env}^{\mathfrak{e}}$ the set of environments under eager semantics.

Eager Denotational Semantics

Intuition

- t: a typable term with type τ
- $[\![t]\!]^{\mathfrak{e}} : \mathbf{Env}^{\mathfrak{e}} \to (V_{\tau}^{\mathfrak{e}})_{\perp}$: the denotational semantics of t

Eager Denotational Semantics

Well-Founded Recursion

- $[\![x]\!]^{\mathfrak{e}} := \lambda\rho.\lfloor\rho(x)\rfloor$
- $[\![n]\!]^{\mathfrak{e}} := \lambda\rho.\lfloor n\rfloor$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![t_1 \ \mathbf{op} \ t_2]\!]^{\mathfrak{e}} := \lambda\rho.([\![t_1]\!]^{\mathfrak{e}}(\rho) \ \mathbf{op}_{\perp} \ [\![t_2]\!]^{\mathfrak{e}}(\rho))$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![\textbf{if } t_0 \textbf{ then } t_1 \textbf{ else } t_2]\!]^{\mathfrak{e}} := \lambda\rho.\mathit{Cond}([\![t_0]\!]^{\mathfrak{e}}(\rho), [\![t_1]\!]^{\mathfrak{e}}(\rho), [\![t_2]\!]^{\mathfrak{e}}(\rho))$
- Conditional (Chapter 9.3):

$$\mathit{Cond}(z_0, z_1, z_2) := \begin{cases} z_1 & \text{if } z_0 = \lfloor 0 \rfloor \\ z_2 & \text{if } z_0 = \lfloor n \rfloor \text{ and } n \neq 0 \\ \bot & \text{if } z_0 = \bot \end{cases}$$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![\textbf{if } t_0 \textbf{ then } t_1 \textbf{ else } t_2]\!]^e := \lambda\rho.Cond([\![t_0]\!]^e(\rho), [\![t_1]\!]^e(\rho), [\![t_2]\!]^e(\rho))$
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-

$$[\![\textbf{if } t_0 \textbf{ then } t_1 \textbf{ else } t_2]\!]^e := \lambda\rho.(let\ n \Leftarrow [\![t_0]\!]^e(\rho).[[\![t_1]\!]^e(\rho), [\![t_2]\!]^e(\rho)](n))$$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![(t_1, t_2)]\!]^{\mathfrak{e}} := \lambda\rho.\mathit{let}\ v_1 \Leftarrow [\![t_1]\!]^{\mathfrak{e}}(\rho), v_2 \Leftarrow [\![t_2]\!]^{\mathfrak{e}}(\rho).\lfloor(v_1, v_2)\rfloor$;
- $[\![\mathbf{fst}(t)]\!]^{\mathfrak{e}} := \lambda\rho.\mathit{let}\ v \Leftarrow [\![t]\!]^{\mathfrak{e}}(\rho).\lfloor\pi_1(v)\rfloor$
- $[\![\mathbf{snd}(t)]\!]^{\mathfrak{e}} := \lambda\rho.\mathit{let}\ v \Leftarrow [\![t]\!]^{\mathfrak{e}}(\rho).\lfloor\pi_2(v)\rfloor$

Eager Denotational Semantics

Function Update (Chapter 9.3)

- $in_1 : \{x\} \to \mathbf{Var}$: $x \mapsto x$
- $in_2 : \mathbf{Var} \setminus \{x\} \to \mathbf{Var}$: $y \mapsto y, y \neq x$

$$\rho\,[v/x] = \lambda y.\mathbf{case}\ y\ \mathbf{of}\ in_1(y_1).v \mid in_2(y_2).\rho(y_2)$$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![\lambda x.t]\!]^{\mathfrak{e}} := \lambda\rho.\lfloor \lambda v \in V^{\mathfrak{e}}_{\tau_1}.[\![t]\!]^{\mathfrak{e}}(\rho\,[v/x])\rfloor$ for $\lambda x.t : \tau_1 \to \tau_2$
- $[\![(t_1\ t_2)]\!]^{\mathfrak{e}} := \lambda\rho.\mathit{let}\ F \Leftarrow [\![t_1]\!]^{\mathfrak{e}}(\rho), v \Leftarrow [\![t_2]\!]^{\mathfrak{e}}(\rho).F(v)$

Eager Denotational Semantics

Well-Founded Recursion

- $[\![\textbf{let } x \Leftarrow t_1 \textbf{ in } t_2]\!]^{\mathfrak{e}} := \lambda\rho.let\ v \Leftarrow [\![t_1]\!]^{\mathfrak{e}}(\rho).[\![t_2]\!]^{\mathfrak{e}}(\rho\,[v/x])$
- $[\![\textbf{rec}y.(\lambda x.t)]\!]^{\mathfrak{e}} := \lambda\rho.\lfloor\mu F.(\lambda v.[\![t]\!]^{\mathfrak{e}}(\rho[v/x, F/y]))\rfloor$

Eager Denotational Semantics

Lemma

- t: a typable term
- ρ, ρ': two environments such that for all $x \in FV(t)$, we have $\rho(x) = \rho'(x)$

Then we have $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![t]\!]^{\mathfrak{e}}(\rho')$.

Eager Denotational Semantics

Lemma

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Then we have $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![t]\!]^{\mathfrak{e}}(\rho')$.

Proof

A simple structural induction on t.

Eager Denotational Semantics

Substitution Lemma

- s: a typable closed term with type τ
- x: a variable with type τ
- t: a typable term with type τ'

If $[\![s]\!]^{\mathfrak{e}}(\rho) = \lfloor v \rfloor$, then we have that:

- $t\,[s/x] : \tau'$;
- $[\![t\,[s/x]]\!]^{\mathfrak{e}}(\rho) = [\![t]\!]^{\mathfrak{e}}(\rho\,[v/x])$.

Eager Denotational Semantics

Substitution Lemma

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If $[\![s]\!]^{\mathfrak{e}}(\rho) = \lfloor v \rfloor$, then we have that:

- $t\,[s/x] : \tau'$;
- $[\![t\,[s/x]]\!]^{\mathfrak{e}}(\rho) = [\![t]\!]^{\mathfrak{e}}(\rho\,[v/x])$.

Proof

By structural induction on t.

Eager Denotational Semantics

Lemma

- If $t : \tau$, then for all ρ we have $[\![t]\!]^{\mathfrak{e}}(\rho) \in (V_\tau^{\mathfrak{e}})_\perp$.
- If $c \in C_\tau^{\mathfrak{e}}$, then for all ρ we have $[\![c]\!]^{\mathfrak{e}}(\rho) \neq \perp$ (the bottom element of $(V_\tau^{\mathfrak{e}})_\perp$).

Eager Denotational Semantics

Lemma

- If $t : \tau$, then for all ρ we have $[\![t]\!]^{\mathfrak{e}}(\rho) \in (V_\tau^{\mathfrak{e}})_\perp$.
- If $c \in C_\tau^{\mathfrak{e}}$, then for all ρ we have $[\![c]\!]^{\mathfrak{e}}(\rho) \neq \perp$ (the bottom element of $(V_\tau^{\mathfrak{e}})_\perp$).

Proof

By structural induction.

Agreement of Eager Semantics

Textbook, Chapter 11.4

Agreement of Eager Semantics

A First Statement

- t: a closed typable term
- c: a canonical term

Then we may expect that $t \to^{\mathfrak{e}} c$ iff $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$.

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A Problem

- t: a closed typable term
- c: a canonical term

$[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho) \Rightarrow t \to^{\mathfrak{e}} c$ may not hold.

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The Correct Theorem

- t: a closed typable term
- c: a canonical term

Then we have:

- $t \to^{\mathfrak{e}} c$ implies $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$;
- The two eager semantics agree on the convergence of t.

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Operational Convergence

- t: a typable closed term

We say that t is *operationally convergent*, denoted by $t{\downarrow^{\mathfrak{e}}}$, if it holds that $\exists c.t \to^{\mathfrak{e}} c$.

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Denotational Convergence

- t: a typable closed term with type τ

We say that t is *denotationally convergent*, denoted by $t{\Downarrow^{\mathfrak{e}}}$, if it holds that $\exists v \in V_\tau^{\mathfrak{e}}.[\![t]\!]^{\mathfrak{e}}(\rho) = \lfloor v \rfloor$.

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The Correct Theorem

- t: a closed typable term
- c: a canonical term

Then we have:

- $t \to^{\mathfrak{e}} c$ implies $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$;
- $t{\downarrow^{\mathfrak{e}}}$ iff $t{\Downarrow^{\mathfrak{e}}}$.

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A Corollary

- t: a closed typable term with type **int**

Then we have that $t \to^{\mathfrak{e}} n$ iff $[\![t]\!]^{\mathfrak{e}}(\rho) = \lfloor n \rfloor$.

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The Correct Theorem

- t: a closed typable term
- c: a canonical term

Then we have:

- $t \to^{\mathfrak{e}} c$ implies $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$;
- $t{\downarrow}^{\mathfrak{e}}$ iff $t{\Downarrow}^{\mathfrak{e}}$.

Main Task

How can we prove this theorem?

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Lemma

- t: a closed typable term
- c: a canonical term

If $t \to^{\mathfrak{e}} c$ then $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$ (for any environment ρ).

Proof.

By rule induction on evaluation of terms. □

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The Rule for **fst**$(-)$

$$\frac{t \to^{\mathfrak{e}} (c_1,\ c_2)}{\mathbf{fst}(t) \to^{\mathfrak{e}} c_1}$$

- $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![(c_1,\ c_2)]\!]^{\mathfrak{e}}(\rho)$;
- $[\![(c_1,\ c_2)]\!]^{\mathfrak{e}}(\rho) = \mathit{let}\ v_1 \Leftarrow [\![c_1]\!]^{\mathfrak{e}}(\rho), v_2 \Leftarrow [\![c_2]\!]^{\mathfrak{e}}(\rho).\lfloor (v_1, v_2) \rfloor$;
- $(c_1,\ c_2)\Downarrow^{\mathfrak{e}}$;
- $\lfloor v_1 \rfloor = [\![c_1]\!]^{\mathfrak{e}}(\rho)$ and $\lfloor v_2 \rfloor = [\![c_2]\!]^{\mathfrak{e}}(\rho)$;
- $[\![\mathbf{fst}(t)]\!]^{\mathfrak{e}}(\rho) = \mathit{let}\ v \Leftarrow [\![t]\!]^{\mathfrak{e}}(\rho).\lfloor \pi_1(v) \rfloor = \lfloor v_1 \rfloor = [\![c_1]\!]^{\mathfrak{e}}(\rho)$.

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The Rule for Function Application

$$\frac{t_1 \to^{\mathfrak{e}} \lambda x.t_1',\ t_2 \to^{\mathfrak{e}} c_2,\ t_1'[c_2/x] \to^{\mathfrak{e}} c}{(t_1\ t_2) \to^{\mathfrak{e}} c}$$

- $[\![t_1]\!]^{\mathfrak{e}}(\rho) = [\![\lambda x.t_1']\!]^{\mathfrak{e}}(\rho)$, $[\![t_2]\!]^{\mathfrak{e}}(\rho) = [\![c_2]\!]^{\mathfrak{e}}(\rho)$ and $[\![t_1'[c_2/x]]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$;

$$\begin{aligned}
[\![(t_1\ t_2)]\!]^{\mathfrak{e}}(\rho) &= \mathit{let}\ F \Leftarrow [\![t_1]\!]^{\mathfrak{e}}(\rho), v \Leftarrow [\![t_2]\!]^{\mathfrak{e}}(\rho).F(v) \\
&= \mathit{let}\ F \Leftarrow [\![\lambda x.t_1']\!]^{\mathfrak{e}}(\rho), v \Leftarrow [\![t_2]\!]^{\mathfrak{e}}(\rho).F(v) \\
&= \mathit{let}\ F \Leftarrow \lfloor \lambda v.[\![t_1']\!]^{\mathfrak{e}}(\rho[v/x]) \rfloor, v \Leftarrow [\![c_2]\!]^{\mathfrak{e}}(\rho).F(v) \\
&= [\![t_1']\!]^{\mathfrak{e}}(\rho[v/x]) \text{ where } \lfloor v \rfloor = [\![c_2]\!]^{\mathfrak{e}}(\rho) \\
&= [\![t_1'[c_2/x]]\!]^{\mathfrak{e}}(\rho) \\
&= [\![c]\!]^{\mathfrak{e}}(\rho)
\end{aligned}$$

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The Rule for Recursion

$$\frac{}{\mathbf{rec}y.(\lambda x.t) \ \to^{\mathfrak{e}} \ \lambda x.(t\,[\mathbf{rec}y.(\lambda x.t)/y])}$$

- $[\![\mathbf{rec}y.(\lambda x.t)]\!]^{\mathfrak{e}}(\rho) = \lfloor F^* \rfloor$;
- $F^* = \mu F.(\lambda v.[\![t]\!]^{\mathfrak{e}}(\rho[v/x, F/y]))$;

$$\begin{aligned}
[\![\lambda x.t\,[\mathbf{rec}y.(\lambda x.t)/y]]\!]^{\mathfrak{e}}(\rho) &= [\![\lambda x.t]\!]^{\mathfrak{e}}(\rho\,[F^*/y]) \\
&= \lfloor \lambda v.[\![t]\!]^{\mathfrak{e}}(\rho[v/x, F^*/y]) \rfloor \\
&= \lfloor F^* \rfloor \\
&= [\![\mathbf{rec}y.(\lambda x.t)]\!]^{\mathfrak{e}}(\rho)
\end{aligned}$$

Agreement of Eager Semantics

Lemma

- t: a closed typable term
- c: a canonical term

If $t \to^{\mathfrak{e}} c$ then $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$ (for any environment ρ).

Corollary

- t: a closed typable term

Then we have that $t{\downarrow}^{\mathfrak{e}} \Rightarrow t{\Downarrow}^{\mathfrak{e}}$.

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The Difficult Part

- t: a closed typable term

Then we have that $t{\Downarrow^{\mathfrak{e}}} \Rightarrow t{\downarrow^{\mathfrak{e}}}$.

The First Attempt

Structural induction on t.

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The Case $t = (t_1\ t_2)$

- induction hypothesis: $(t_1{\Downarrow^e} \Rightarrow t_1{\downarrow^e})$ & $(t_2{\Downarrow^e} \Rightarrow t_2{\downarrow^e})$

Then

- $t{\Downarrow^e}$
- $[\![t]\!]^e(\rho) = \mathit{let}\ F \Leftarrow [\![t_1]\!]^e(\rho), v \Leftarrow [\![t_2]\!]^e(\rho).F(v)$
- $t_1{\Downarrow^e}$ & $t_2{\Downarrow^e}$
- $t_1 \to^e \lambda x.t_1'$ & $t_2 \to^e c_2$
- $[\![t]\!]^e(\rho) = F(v)$ where $F = \lambda u.[\![t_1']\!]^e(\rho\,[u/x])$ and $\lfloor v \rfloor = [\![c_2]\!]^e(\rho)$
- $[\![t]\!]^e(\rho) = [\![t_1']\!]^e(\rho\,[v/x]) = [\![t_1'\,[c_2/x]]\!]^e(\rho)$
- $t_1'\,[c_2/x]{\downarrow^e}$ and $t_1'\,[c_2/x] \to^e c$
- $t \to^e c$

Agreement of Eager Semantics

Question

What's wrong with the proof?

Agreement of Eager Semantics

Question

What's wrong with the proof?

Proof

- ...
- $t_1 \to^{\mathfrak{e}} \lambda x.t_1' \ \&\ t_2 \to^{\mathfrak{e}} c_2$
- ...
- $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![t_1']\!]^{\mathfrak{e}}(\rho\,[v/x]) = [\![t_1'\,[c_2/x]]\!]^{\mathfrak{e}}(\rho)$
- $t_1'\,[c_2/x]{\downarrow^{\mathfrak{e}}}$ and $t_1'\,[c_2/x] \to^{\mathfrak{e}} c$??
- $t \to^{\mathfrak{e}} c$

Agreement of Eager Semantics

Question

What's wrong with the proof?

The Problem

- It is not guaranteed that $t_1' [c_2/x]\Downarrow^{\mathfrak{e}} \Rightarrow t_1' [c_2/x]\downarrow^{\mathfrak{e}}$.

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Solution

- a stronger induction hypothesis
- a *logical relation* between values and types

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Logical Relations

- τ: a type

Then we will define:

- $\lesssim_\tau^\circ \subseteq V_\tau^{\mathfrak{e}} \times C_\tau^{\mathfrak{e}}$
- $\lesssim_\tau \subseteq (V_\tau^{\mathfrak{e}})_\perp \times \mathit{ClosedTerms}$

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The Relation $\lesssim_\tau$

- τ: a type

We define the relation $\lesssim_\tau \subseteq (V_\tau^{\mathfrak{e}})_\perp \times \mathit{ClosedTerms}$ by:

$$d \lesssim_\tau t \text{ iff } \forall v \in V_\tau^{\mathfrak{e}}.\, [d = \lfloor v \rfloor \Rightarrow (\exists c.(t \to^{\mathfrak{e}} c \;\&\; v \lesssim_\tau^{\circ} c))]$$

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The Relations $\lesssim^{\circ}_{\tau} \subseteq V^{\mathfrak{e}}_{\tau} \times C^{\mathfrak{e}}_{\tau}$

- ground types:

$$n \lesssim^{\circ}_{\mathbf{int}} n \text{ for all integers } n$$

- product types:

$$(v_1, v_2) \lesssim^{\circ}_{\tau_1 * \tau_2} (c_1, c_2) \text{ iff } v_1 \lesssim^{\circ}_{\tau_1} c_1 \text{ and } v_2 \lesssim^{\circ}_{\tau_2} c_2$$

- function types:

$$F \lesssim^{\circ}_{\tau_1 \to \tau_2} \lambda x.t \text{ iff } \forall v \in V^{\mathfrak{e}}_{\tau_1}, c \in C^{\mathfrak{e}}_{\tau_2}. v \lesssim^{\circ}_{\tau_1} c \Rightarrow F(v) \lesssim_{\tau_2} t\,[c/x]$$

Agreement of Eager Semantics

Lemma

- t: a closed term with type τ

We have that

- $\bot_{(V_\tau^{\mathfrak{e}})_\bot} \lesssim_\tau t$;
- for any $d, d' \in (V_\tau^{\mathfrak{e}})_\bot$, it holds that

$$\left(d \sqsubseteq d' \;\&\; d' \lesssim_\tau t \Rightarrow d \lesssim_\tau t\right);$$

- for any $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in $(V_\tau^{\mathfrak{e}})_\bot$, it holds that

$$(\forall n. d_n \lesssim_\tau t) \Rightarrow \bigsqcup_{n\in\omega} d_n \lesssim_\tau t.$$

Agreement of Eager Semantics

Lemma

- $\bot_{(V_\tau^{\mathfrak{e}})_\bot} \lesssim_\tau t$;

Proof

By definition:

$$d \lesssim_\tau t \text{ iff } \forall v \in V_\tau^{\mathfrak{e}}. [d = \lfloor v \rfloor \Rightarrow (\exists c.(t \to^{\mathfrak{e}} c \ \& \ v \lesssim_\tau^\circ c))]$$

Agreement of Eager Semantics

Lemma

- for any $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in $(V_\tau^{\mathfrak{e}})_\perp$, it holds that

$$(\forall n. d_n \lesssim_\tau t) \Rightarrow \bigsqcup_{n \in \omega} d_n \lesssim_\tau t.$$

Proof (Structural Induction on Types)

base type: $\tau = \textbf{int}$. Straightforward.

Agreement of Eager Semantics

Lemma

- for any $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in $(V^{\mathfrak{e}}_{\tau})_{\perp}$, it holds that

$$(\forall n. d_n \lesssim_{\tau} t) \Rightarrow \bigsqcup_{n \in \omega} d_n \lesssim_{\tau} t.$$

Proof (Structural Induction on Types)

function types: $\tau = \tau_1 \to \tau_2$.

- Suppose $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \ldots$ in $(V^{\mathfrak{e}}_{\tau_1 \to \tau_2})_{\perp}$.
- Suppose that $\forall n. d_n \lesssim_{\tau_1 \to \tau_2} t$.
- easy case: $\forall n. d_n = \perp$

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Proof (Structural Induction on Types)

function types: $\tau = \tau_1 \to \tau_2$.

- Suppose that $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $(V^{\mathfrak{e}}_{\tau_1 \to \tau_2})_\perp$ and $\forall n. d_n \lesssim_{\tau_1 \to \tau_2} t$.
- nontrivial case: $\exists n. d_n \neq \perp$ and $t \to^{\mathfrak{e}} \lambda x. t'$.
- from definition: $\forall n \geq N. (d_n = \lfloor F_n \rfloor \ \& \ F_n \lesssim^{\circ}_{\tau_1 \to \tau_2} \lambda x. t')$
- $\forall n \geq N. \forall (v, c) \in V^{\mathfrak{e}}_{\tau_1} \times C^{\mathfrak{e}}_{\tau_1}. v \lesssim^{\circ}_{\tau_1} c \Rightarrow F_n(v) \lesssim_{\tau_2} t'[c/x]$
- induction hypothesis: $(\bigsqcup_n F_n)(v) = \bigsqcup_n (F_n(v)) \lesssim_{\tau_2} t'[c/x]$
- $\forall (v, c). v \lesssim^{\circ}_{\tau_1} c \Rightarrow (\bigsqcup_n F_n)(v) \lesssim_{\tau_2} t'[c/x]$
- $\bigsqcup_{n \in \omega} F_n \lesssim^{\circ}_{\tau_1 \to \tau_2} \lambda x. t'$
- $\bigsqcup_{n \in \omega} d_n = \lfloor \bigsqcup_{n \in \omega} F_n \rfloor \lesssim_{\tau_1 \to \tau_2} t$

Agreement of Eager Semantics

Lemma

- t: a typable close term

We have that $t\Downarrow^{\mathfrak{e}} \Rightarrow t\downarrow^{\mathfrak{e}}$.

Proof (by Structural Induction)

We prove by structural induction on terms that:

- $t : \tau$: a term
- $x_1 : \tau_1, \ldots, x_k : \tau_k$: free variables in t
- $s_1 : \tau_1, \ldots, s_k : \tau_k$: closed terms
- $v_i \in V^{\mathfrak{e}}_{\tau_i}$ $(1 \leq i \leq k)$: elements such that $\lfloor v_i \rfloor \lesssim_{\tau_i} s_i$

Then $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

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- $v_i \in V^{\mathfrak{e}}_{\tau_i}$ $(1 \leq i \leq k)$: elements such that $\lfloor v_i \rfloor \lesssim_{\tau_i} s_i$

Then $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $t : \tau$: a term
- $x_1 : \tau_1, \ldots, x_k : \tau_k$: free variables in t
- $s_1 : \tau_1, \ldots, s_k : \tau_k$: closed terms
- $v_i \in V^{\mathfrak{e}}_{\tau_i}$ $(1 \leq i \leq k)$: elements such that $\lfloor v_i \rfloor \lesssim_{\tau_i} s_i$

Then $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Base Step: $t = x$ and $x : \tau$

- Suppose $\lfloor v \rfloor \lesssim_{\tau} s$.
- $[\![x]\!]^{\mathfrak{e}}(\rho\,[v/x]) = \lfloor v \rfloor \lesssim_{\tau} s = x\,[s/x]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $t : \tau$: a term
- $x_1 : \tau_1, \ldots, x_k : \tau_k$: free variables in t
- $s_1 : \tau_1, \ldots, s_k : \tau_k$: closed terms
- $v_i \in V^{\mathfrak{e}}_{\tau_i}$ $(1 \leq i \leq k)$: elements such that $\lfloor v_i \rfloor \lesssim_{\tau_i} s_i$

Then $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Base Step: $t = n$

- $n \lesssim^{\circ}_{\mathbf{int}} n$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = t_1$ **op** t_2

- Suppose that $[\![t_1 \textbf{ op } t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor n \rfloor$.
- Then
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor n_1 \rfloor$.
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor n_2 \rfloor$.
  - $n = n_1$ **op** n_2.
- By induction hypothesis,
  - $\lfloor n_1 \rfloor \lesssim_{\textbf{int}} t_1[v_1/x_1, \ldots, v_k/x_k]$
  - $\lfloor n_2 \rfloor \lesssim_{\textbf{int}} t_2[v_1/x_1, \ldots, v_k/x_k]$

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = t_1$ **op** t_2

- By induction hypothesis,
  - $\lfloor n_1 \rfloor \lesssim_{\mathbf{int}} t_1[v_1/x_1, \ldots, v_k/x_k]$
  - $\lfloor n_2 \rfloor \lesssim_{\mathbf{int}} t_2[v_1/x_1, \ldots, v_k/x_k]$
- From the definition of $\lesssim_{\mathbf{int}}$,
  - $t_1[v_1/x_1, \ldots, v_k/x_k] \to^{\mathfrak{e}} n_1$
  - $t_2[v_1/x_1, \ldots, v_k/x_k] \to^{\mathfrak{e}} n_2$
- Hence, $t[v_1/x_1, \ldots, v_k/x_k] \to^{\mathfrak{e}} n$.
- Finally, $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t =$ **if** t_0 **then** t_1 **else** t_2

- Suppose that $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$.
- Then either
  - $[\![t_0]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor 0 \rfloor$.
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_1 \rfloor$.

  or
  - $[\![t_0]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor n \rfloor$ $(n > 0)$.
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_2 \rfloor$.
- ...

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t =$ **if** t_0 **then** t_1 **else** t_2

- Suppose that $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$.
- Then either
  - $[\![t_0]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor 0 \rfloor$.
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_1 \rfloor$.

  or
  - $[\![t_0]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor n \rfloor$ $(n > 0)$.
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_2 \rfloor$.
- ...

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = (t_1, t_2)$

- Suppose $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$ and $t_1 : \tau_1, t_2 : \tau_2$.
- Then
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_1 \rfloor$.
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_2 \rfloor$.
  - $u = (u_1, u_2)$.
- By induction hypothesis,
  - $\lfloor u_1 \rfloor \lesssim_{\tau_1} t_1[s_1/x_1, \ldots, s_k/x_k]$;
  - $\lfloor u_2 \rfloor \lesssim_{\tau_2} t_2[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = (t_1, t_2)$

- By induction hypothesis,
  - $\lfloor u_1 \rfloor \lesssim_{\tau_1} t_1[s_1/x_1, \ldots, s_k/x_k]$;
  - $\lfloor u_2 \rfloor \lesssim_{\tau_2} t_2[s_1/x_1, \ldots, s_k/x_k]$.
- By definition,
  - $t_1[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c_1$ and $u_1 \lesssim^{\circ}_{\tau_1} c_1$;
  - $t_2[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c_2$ and $u_2 \lesssim^{\circ}_{\tau_2} c_2$.
- Then $(u_1, u_2) \lesssim^{\circ}_{\tau} (c_1, c_2)$ and $t[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} (c_1, c_2)$.
- Finally, $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \mathbf{fst}(t')$

- Suppose that $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$ and $t' : \tau_1 * \tau_2$;
- Then $[\![t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor (u_1, u_2) \rfloor$ and $u = u_1$;
- By induction hypothesis, $\lfloor (u_1, u_2) \rfloor \lesssim_{\tau_1 * \tau_2} t'[s_1/x_1, \ldots, s_k/x_k]$;
- Hence, $t'[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} (c_1, c_2)$ and $(u_1, u_2) \lesssim^{\circ}_{\tau_1 * \tau_2} (c_1, c_2)$;
- Then, $t[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c_1$ and $u_1 \lesssim^{\circ}_{\tau_1} c_1$;
- Finally, $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_{\tau} t[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \mathbf{snd}(t')$

By similar proof.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \lambda x.t'$ with $x : \tau$, $t' : \tau'$

- $[\![\lambda x.t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor F \rfloor$.
- Then $F = \lambda v \in V_\tau^{\mathfrak{e}}.[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k, v/x])$
- For any v, c such that $v \lesssim_\tau^\circ c$, $\lfloor v \rfloor \lesssim_\tau c$ and $F(v) = [\![t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k, v/x])$
- By induction hypothesis,

$$[\![t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k, v/x]) \lesssim_{\tau'} t'[s_1/x_1, \ldots, s_k/x_k, c/x]$$

- $F \lesssim^\circ_{\tau \to \tau'} \lambda x.t'[s_1/x_1, \ldots, s_k/x_k]$
- $\lfloor F \rfloor \lesssim^\circ_{\tau \to \tau'} t[s_1/x_1, \ldots, s_k/x_k]$

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = (t_1\ t_2)$ with $t_1 : \tau' \to \tau$, $t_2 : \tau'$

- Suppose that $[\![(t_1\ t_2)]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$.
- Then we have
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor F \rfloor$.
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor v \rfloor$.
  - $F(v) = \lfloor u \rfloor$.
- By induction hypothesis,
  - $\lfloor F \rfloor \lesssim_{\tau' \to \tau} t_1[s_1/x_1, \ldots, s_k/x_k]$.
  - $\lfloor v \rfloor \lesssim_{\tau'} t_2[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = (t_1\ t_2)$ with $t_1 : \tau' \to \tau''$, $t_2 : \tau'$

- By induction hypothesis,
  - $\lfloor F \rfloor \lesssim_{\tau' \to \tau} t_1[s_1/x_1, \ldots, s_k/x_k]$.
  - $\lfloor v \rfloor \lesssim_{\tau'} t_2[s_1/x_1, \ldots, s_k/x_k]$.
- By definition,
  - $t_1[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} \lambda x.t_1'$ and $F \lesssim^{\circ}_{\tau' \to \tau} \lambda x.t_1'$;
  - $t_2[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c_2$ and $v \lesssim^{\circ}_{\tau'} c_2$.
- From $F \lesssim^{\circ}_{\tau' \to \tau} \lambda x.t_1'$, we have $\lfloor u \rfloor = F(v) \lesssim_\tau t_1'\,[c_2/x]$.
- Then, there is $c \in C^{\mathfrak{e}}_\tau$ such that $t_1'\,[c_2/x] \to^{\mathfrak{e}} c$ and $u \lesssim^{\circ}_\tau c$;
- $(t_1\ t_2)[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c$;
- Finally, $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \mathit{let}\ x \Leftarrow t_1.t_2$ with $x : \tau_1,\ t_1 : \tau_1,\ t_2 : \tau$

- Suppose that $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u \rfloor$
- Then there is $u_1 \in V^{\mathfrak{e}}_{\tau_1}$ such that
  - $[\![t_1]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor u_1 \rfloor$;
  - $[\![t_2]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k][u_1/x]) = \lfloor u \rfloor$;
- From induction hypothesis, there are canonical forms c_1, c such that
  - $u_1 \lesssim^{\circ}_{\tau_1} c_1$ and $t_1[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c_1$
  - $u \lesssim^{\circ}_{\tau_1} c$ and $t_2[s_1/x_1, \ldots, s_k/x_k][c_1/x] \to^{\mathfrak{e}} c$
- Thus, $t[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} c$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \mathbf{rec}y.(\lambda x.t')$ with $x : \tau''$, $t' : \tau'$

- Suppose that $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) = \lfloor G \rfloor$ for $G \in V^{\mathfrak{e}}_{\tau'' \to \tau'}$
- $G = \mu F.(\lambda v.[\![t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k, v/x, F/y]))$
- $G = \bigsqcup_{n\in\omega} G_n = \bigcup_{n\in\omega} G_n$ where
  - $G_0 := \bot_{V^{\mathfrak{e}}_{\tau'' \to \tau'}}$
  - $G_{n+1} := \lambda v.[\![t']\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k, v/x, G_n/y])$
- By induction: for all n, $G_n \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$
- Then $G \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$.
- Note that $t[s_1/x_1, \ldots, s_k/x_k] \to^{\mathfrak{e}} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$.
- Thus, $\lfloor G \rfloor \lesssim^{\circ}_{\tau'' \to \tau'} t[s_1/x_1, \ldots, s_k/x_k]$.

Agreement of Eager Semantics

Proof (by Structural Induction)

- $[\![t]\!]^{\mathfrak{e}}(\rho[v_1/x_1, \ldots, v_k/x_k]) \lesssim_\tau t[s_1/x_1, \ldots, s_k/x_k]$.

Inductive Step: $t = \mathbf{rec}y.(\lambda x.t')$ with $x : \tau''$, $t' : \tau'$

- $G_0 := \bot_{V^{\mathfrak{e}}_{\tau'' \to \tau'}}$ and $G_{n+1} := \lambda v.[\![t']\!]^{\mathfrak{e}}(\rho[\vec{\mathbf{v}}/\vec{\mathbf{x}}, v/x, G_n/y])$
- By induction: for all n, $G_n \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$
- Base Step: $\bot_{V^{\mathfrak{e}}_{\tau'' \to \tau'}} \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$
- For all v, c such that $v \lesssim^{\circ}_{\tau''} c$,

$$\bot_{V^{\mathfrak{e}}_{\tau'' \to \tau'}}(v) \lesssim_{\tau'} t'[\mathbf{s}/\mathbf{x}, c/x, t[\mathbf{s}/\mathbf{x}]/y].$$

Agreement of Eager Semantics

Inductive Step: $t = \mathbf{rec}y.(\lambda x.t')$ with $x : \tau''$, $t' : \tau'$

- $G_0 := \bot_{V^{\mathfrak{e}}_{\tau'' \to \tau'}}$ and $G_{n+1} := \lambda v.[\![t']\!]^{\mathfrak{e}}(\rho[\vec{\mathbf{v}}/\vec{\mathbf{x}}, v/x, G_n/y])$
- By induction: for all n, $G_n \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$
- Inductive Step: suppose $G_n \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$.
- Then $\lfloor G_n \rfloor \lesssim_{\tau'' \to \tau'} t[\mathbf{s}/\mathbf{x}]$.
- Consider any $v \lesssim^{\circ}_{\tau''} c$ so that $\lfloor v \rfloor \lesssim_{\tau''} c$.
- Note that $G_{n+1}(v) = [\![t']\!]^{\mathfrak{e}}(\rho[\vec{\mathbf{v}}/\vec{\mathbf{x}}, v/x, G_n/y])$.
- By the main induction hypothesis,

$$G_{n+1}(v) \lesssim_{\tau'} t'[\mathbf{s}/\mathbf{x}, c/x, t[\mathbf{s}/\mathbf{x}]/y].$$

- $G_{n+1} \lesssim^{\circ}_{\tau'' \to \tau'} (\lambda x.t')[\mathbf{s}/\mathbf{x}, t[\mathbf{s}/\mathbf{x}]/y]$.

Agreement of Eager Semantics

Corollary

- $t : \mathbf{int}$: a closed term

Then $t \to^{\mathfrak{e}} n$ iff $[\![t]\!]^{\mathfrak{e}}(\rho) = \lfloor n \rfloor$.

Lazy Denotational Semantics

Textbook, Chapter 11.7

Lazy Denotational Semantics

Values

- τ: a type

The discrete cpo $V_\tau^\mathfrak{l}$ of values associated with type τ is recursively defined as follows:

- $V_{\mathbf{int}}^\mathfrak{l} := \mathbb{Z}$;
- $V_{\tau_1 * \tau_2}^\mathfrak{l} := (V_{\tau_1}^\mathfrak{l})_\perp \times (V_{\tau_2}^\mathfrak{l})_\perp$;
- $V_{\tau_1 \to \tau_2}^\mathfrak{l} := [(V_{\tau_1}^\mathfrak{l})_\perp \to (V_{\tau_2}^\mathfrak{l})_\perp]$

Lazy Denotational Semantics

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- $V_{\tau_1 \to \tau_2}^{\mathfrak{l}} := [(V_{\tau_1}^{\mathfrak{l}})_\perp \to (V_{\tau_2}^{\mathfrak{l}})_\perp]$

Question

Why do we have extra $\perp$'s?

Lazy Denotational Semantics

Environments

- **Var**: the set of variables

An *environment* ρ is a function

$$\rho : \mathbf{Var} \to \bigcup \{(V_\tau^{\mathfrak{l}})_\perp \mid \tau \text{ a type}\}$$

such that

$$\forall x \in \mathbf{Var}.(x : \tau \Rightarrow \rho(x) \in (V_\tau^{\mathfrak{l}})_\perp) \ .$$

Lazy Denotational Semantics

Environments

- **Var**: the set of variables

An *environment* ρ is a function

$$\rho : \mathbf{Var} \to \bigcup \{(V_\tau^{\mathfrak{l}})_\perp \mid \tau \text{ a type}\}$$

such that

$$\forall x \in \mathbf{Var}.(x : \tau \Rightarrow \rho(x) \in (V_\tau^{\mathfrak{l}})_\perp) \ .$$

We denote by $\mathbf{Env}^{\mathfrak{l}}$ the set of environments under lazy semantics.

Lazy Denotational Semantics

Intuition

- t: a typable term with type τ
- $[\![t]\!]^{\Gamma} : \mathbf{Env}^{\Gamma} \to (V_{\tau}^{\Gamma})_{\perp}$: the denotational semantics of t

Lazy Denotational Semantics

Inductive Definition

- $[\![x]\!]^{\mathfrak{l}} := \lambda\rho.\rho(x)$;
- $[\![n]\!]^{\mathfrak{l}} := \lambda\rho.\lfloor n \rfloor$;

Lazy Denotational Semantics

Inductive Definition

- $[\![t_1 \ \mathbf{op} \ t_2]\!]^{\mathfrak{l}} := \lambda\rho.([\![t_1]\!]^{\mathfrak{l}}(\rho) \ \mathbf{op}_{\perp} \ [\![t_2]\!]^{\mathfrak{l}}(\rho))$;
- $$[\![\mathbf{if} \ t_0 \ \mathbf{then} \ t_1 \ \mathbf{else} \ t_2]\!]^{\mathfrak{l}} := \lambda\rho.\mathit{Cond}([\![t_0]\!]^{\mathfrak{l}}(\rho), [\![t_1]\!]^{\mathfrak{l}}(\rho), [\![t_2]\!]^{\mathfrak{l}}(\rho));$$

Lazy Denotational Semantics

Inductive Definition

- $[\![(t_1, t_2)]\!]^{\mathsf{l}} := \lambda\rho.\lfloor([\![t_1]\!]^{\mathsf{l}}(\rho), [\![t_2]\!]^{\mathsf{l}}(\rho))\rfloor$;
- $[\![\mathbf{fst}(t)]\!]^{\mathsf{l}} := \lambda\rho.\mathit{let}\ v \Leftarrow [\![t]\!]^{\mathsf{l}}(\rho).\pi_1(v)$
- $[\![\mathbf{snd}(t)]\!]^{\mathsf{l}} := \lambda\rho.\mathit{let}\ v \Leftarrow [\![t]\!]^{\mathsf{l}}(\rho).\pi_2(v)$

Lazy Denotational Semantics

Inductive Definition

- $[\![\lambda x.t]\!]^{\mathfrak{l}} := \lambda\rho.\lfloor\lambda v \in (V^{\mathfrak{l}}_{\tau_1})_{\perp}.[\![t]\!]^{\mathfrak{l}}(\rho\,[v/x])\rfloor$ for $\lambda x.t : \tau_1 \to \tau_2$
- $[\![(t_1\ t_2)]\!]^{\mathfrak{l}} := \lambda\rho.let\ F \Leftarrow [\![t_1]\!]^{\mathfrak{l}}(\rho).F([\![t_2]\!]^{\mathfrak{l}}(\rho))$

Lazy Denotational Semantics

Inductive Definition

- $[\![\textbf{let } x \Leftarrow t_1 \textbf{ in } t_2]\!]^{\mathfrak{l}} := \lambda\rho.[\![t_2]\!]^{\mathfrak{l}}\left(\rho\left[[\![t_1]\!]^{\mathfrak{l}}(\rho)/x\right]\right)$
- $[\![\textbf{rec}y.t]\!]^{\mathfrak{l}} := \lambda\rho.(\mu F.[\![t]\!]^{\mathfrak{l}}(\rho[F/y]))$

Agreement of Lazy Semantics

Operational Convergence

- t: a typable closed term

We say that t is *operationally convergent*, denoted by $t{\downarrow^{\mathsf{l}}}$, if it holds that $\exists c.t \rightarrow^{\mathsf{l}} c$.

Agreement of Lazy Semantics

Denotational Convergence

- t: a typable closed term with type τ

We say that t is *denotationally convergent*, denoted by $t\Downarrow^{\mathfrak{l}}$, if it holds that $\exists v \in V_\tau^{\mathfrak{l}}.[\![t]\!]^{\mathfrak{l}}(\rho) = \lfloor v \rfloor$.

Agreement of Lazy Semantics

The Theorem

- t: a closed typable term
- c: a canonical term

Then we have:

- $t \to^{\mathfrak{e}} c$ implies $[\![t]\!]^{\mathfrak{e}}(\rho) = [\![c]\!]^{\mathfrak{e}}(\rho)$;
- $t{\downarrow^{\mathfrak{e}}}$ iff $t{\Downarrow^{\mathfrak{e}}}$.

Agreement of Lazy Semantics

A Corollary

- t: a closed typable term with type **int**

Then we have that $t \to^{\mathfrak{l}} n$ iff $[\![t]\!]^{\mathfrak{l}}(\rho) = \lfloor n \rfloor$.

Summary

- eager denotational semantics
- lazy denotational semantics
- agreement of the semantics

No Homework This Time!