Basic Set Theory

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Topics

An Informal Introduction on Set Theory

- What are sets?
- How can one reason about sets?
- How can one construct sets?
- How relations and functions are defined in set theory?

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- How can one construct sets?
- How relations and functions are defined in set theory?

Textbook Content

- Chapter 1: Basic Set Theory

Set Theory: An Intuitive Description

Why Set Theory?

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- a rigorous language for a logical world

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- a rigorous language for a logical world
- a solid foundation for mathematics

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- a rigorous language for a logical world
- a solid foundation for mathematics
- a solid foundation for programming languages

Why Set Theory?

- Reasoning without a solid foundation is error-prone.

Why Set Theory?

- Reasoning without a solid foundation is error-prone.
- Reasoning with a solid foundation is precise.

Set Theory

What is a Set?

A set is a collection of objects that acts as a single entity.

Set Theory: An Overview

- an abstract world: a world of sets as objects

Set Theory: An Overview

- set reasoning: for any object a and set X, either $a \in X$ or $a \notin X$ but not both.

- set construction: any set can be constructed from the empty set through a finite number of axioms

Sets: Examples

- $\{a\}$, $\{a, b\}$, $\{a_1, \ldots, a_n\}$
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- $\mathbb{R} \times \mathbb{R}$
- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \ \& \ x \leq b\}$
- $\{p \in \mathbb{N} \mid \forall q, r \in \mathbb{N}. [p = q \cdot r \Rightarrow (q = 1 \vee r = 1)]\}$

The Language of Set Theory

The Language of Set Theory

- a formal language for sets
- a formal language for mathematical objects, i.e., numbers, functions, graphs, ...
  (as they can be defined as sets in set theory)

The Language of Set Theory

- names: $a_0, a_1, \ldots, A_0, A_1, \ldots$
- variables: $x_0, x_1, \ldots$
- logic connectives: $\neg$, $\&$, or , $\Rightarrow$, $\Leftrightarrow$, $\exists$, $\forall$

Formulas (Clauses, Sentences)

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- boolean connectives:
  - if ϕ is a formula, then so is $\neg\phi$;
  - if ϕ_1, ϕ_2 are formulas, then so too are

$$\phi_1 \ \& \ \phi_2 \ , \ \phi_1 \text{ or } \phi_2 \ , \ \phi_1 \Rightarrow \phi_2 \ , \ \phi_1 \Leftrightarrow \phi_2 \ ;$$

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- quantifiers: if ϕ is a formula and x is a variable, then $\forall x.\phi$, $\exists x.\phi$ are formulas.

Atomic Formulae

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Atomic Formulae

- (equality) $x = y$
  (meaning the assertion "x, y name the same object (set)")
- (membership) $x \in y$
  (meaning the assertion "x is an element of y")
- evaluated to
  - either **true** (i.e. the formula holds),
  - or **false** (i.e., the formula does not hold)

  when the meaning of x, y (i.e., which sets x, y name) is clear

Boolean Connectives

- $\neg$: negation ("not")
- $\&$: conjunction ("and")
- or: disjunction ("or")
- $\Rightarrow$: implication ("imply")
- $\Leftrightarrow$: double implication ("iff")

Truth Table

Negation ("not")

ϕ	$\neg\phi$
true	**false**
false	**true**

Truth Table

Conjunction ("and")

ϕ_1	ϕ_2	ϕ_1 & ϕ_2
true	**true**	**true**
true	**false**	**false**
false	**true**	**false**
false	**false**	**false**

Truth Table

Disjunction ("or")

ϕ_1	ϕ_2	ϕ_1 or ϕ_2
true	**true**	**true**
true	**false**	**true**
false	**true**	**true**
false	**false**	**false**

Truth Table

Implication ("imply")

ϕ_1	ϕ_2	$\phi_1 \Rightarrow \phi_2$
true	**true**	**true**
true	**false**	**false**
false	**true**	**true**
false	**false**	**true**

Truth Table

Double Implication ("iff")

ϕ_1	ϕ_2	$\phi_1 \Leftrightarrow \phi_2$
true	**true**	**true**
true	**false**	**false**
false	**true**	**false**
false	**false**	**true**

Boolean Expressibility

Exercise

Prove (through truth table) that the following formulas are logically equivalent:

- ϕ_1 or ϕ_2 and $\neg(\neg\phi_1 \ \& \ \neg\phi_2)$;
- $\phi_1 \Rightarrow \phi_2$ and $(\neg\phi_1)$ or ϕ_2;
- $\phi_1 \Leftrightarrow \phi_2$ and $(\phi_1 \ \& \ \phi_2)$ or $(\neg\phi_1 \ \& \ \neg\phi_2)$.
- $\phi_1 \Leftrightarrow \phi_2$ and $(\phi_1 \Rightarrow \phi_2) \ \& \ (\phi_2 \Rightarrow \phi_1)$

Quantifiers

Universal Quantification

$\forall x.\phi(x)$ holds (or simply written as $\forall x.\phi(x)$) if

- (intuition) for any set x, $\phi(x)$ holds;
- (meaning) $\phi(x)$ is true (i.e., holds) no matter what x names in the universe of all sets;

Quantifiers

Universal Quantification

$\forall x.\phi(x)$ holds (or simply written as $\forall x.\phi(x)$) if

- (intuition) for any set x, $\phi(x)$ holds;
- (meaning) $\phi(x)$ is true (i.e., holds) no matter what x names in the universe of all sets;

Example

- A, B : sets
- A equals B: $\forall x.\,(x \in A \Leftrightarrow x \in B)$

Quantifiers

Existential Quantification

$\exists x.\phi(x)$ holds (or simply written as $\exists x.\phi(x)$) if

- (intuition) there exists a set x such that $\phi(x)$ holds;
- (meaning) there exists a set such that $\phi(x)$ is true (i.e., holds) when x names that set.

Quantifiers

- no clear thinking (e.g., no truth table)
- $\forall x.\phi$ is logically equivalent to $\neg(\exists x.\neg\phi)$.

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Exercise

Prove that $\forall x.\phi$ is logically equivalent to $\neg(\exists x.\neg\phi)$.

Quantifiers

Conditioned Quantifiers

- A: a set
- $\forall x \in A.\phi(x) \ := \ \forall x.(x \in A \Rightarrow \phi(x))$
- $\exists x \in A.\phi(x) \ := \ \exists x \in A.(x \in A \ \& \ \phi(x))$

Quantifiers

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Exercise

Prove that $\forall x \in A.\phi$ is logically equivalent to $\neg(\exists x \in A.\neg\phi)$.

Axioms

- formulas assumed to be correct
- formulas for asserting properties of sets
- formulas for constructing sets

Axioms for Set Reasoning

Set Reasoning

Extensionality Axiom

- statement: $\forall A \forall B. [\forall x. (x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$
- meaning: if two sets A, B have exactly the same members, then they are equal.

Set Inclusion

- definition: Given any two sets A, B, we write

$$A \subseteq B \text{ if } \forall x.(x \in A \Rightarrow x \in B).$$

- property: For any two sets A, B, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Set Reasoning

Set Inclusion

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$$A \subseteq B \text{ if } \forall x.(x \in A \Rightarrow x \in B).$$

- property: For any two sets A, B, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Exercise

Prove from Extensionality Axiom that $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

The Axiom of Foundation (Regularity)

- statement: $\forall A.\,[A \neq \emptyset \Rightarrow \exists B.\,(B \in A \;\&\; B \cap A = \emptyset)]$.
- corollary: for any set A, $A \notin A$.
- corollary: there is no infinite set sequence $A_0, A_1, \cdots$ such that $\cdots \in A_{n+1} \in A_n \in \cdots \in A_1 \in A_0$.

Axioms for Set Construction

Set Construction

Empty Set Axiom

- statement: $\exists B.(\forall x.x \notin B)$;
- uniqueness: $\forall A \forall B.[(\forall x.x \notin A) \ \& \ (\forall x.x \notin B) \Rightarrow A = B]$
- notation: $\emptyset$ is the set without any member.

Set Construction

Empty Set Axiom

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- notation: $\emptyset$ is the set without any member.

Question

Why do we need uniqueness?

Set Construction

Pairing Axiom

- statement: $\forall u.\forall v.\exists B.\,[\forall x.\,(x \in B \Leftrightarrow x = u \text{ or } x = v)]$;
- uniqueness: from Extensionality Axiom
- notation: $B = \{u, v\}$ ($\{u\} := \{u, u\}$).

Set Construction

Pairing Axiom

- ordered pairs: $(x, y) := \{\{x\}, \{x, y\}\}$;
- property: $(x_1, y_1) = (x_2, y_2)$ iff $x_1 = x_2$ and $y_1 = y_2$.

Set Construction

Union Axiom (Preliminary Version)

- statement: $\forall A.\forall B.\exists C.\,[\forall x.\,(x \in C \Leftrightarrow x \in A \text{ or } x \in B)]$;
- uniqueness: from Extensionality Axiom
- notation: $C = A \cup B$

Set Construction

Union Axiom

- statement: $\forall \mathcal{A}. \exists B. [\forall x. (x \in B \Leftrightarrow \exists A \in \mathcal{A}. x \in A)]$;
- uniqueness: from Extensionality Axiom
- notation: $B = \bigcup \mathcal{A}$
- example: $A \cup B = \bigcup \{A, B\}$

Set Construction

Power Set Axiom

- statement: $\forall A.\exists B.[\forall x.x \in B \Leftrightarrow x \subseteq A]$;
- uniqueness: from Extensionality Axiom
- notation: $B = 2^A$, $B = Pow(A)$ (textbook) or informally $B = \{Y \mid Y \subseteq A\}$

Example

$2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.

Set Construction

Subset Axiom

- statement: for any set A, for any sets $t_1, \ldots, t_n$, for any formula $\phi(x, y_1, \ldots, y_n)$, there exists a set B such that

$$\forall x. \left(x \in B \Leftrightarrow x \in A \ \& \ \phi(x, t_1, \ldots, t_n)\right) \ ;$$

- uniqueness: from Extensionality Axiom
- notation: $B = \{x \in A \mid \phi(x, t_1, \ldots, t_n)\}$

Set Construction

Some Set Operations

- intersection: $A \cap B := \{x \in A \cup B \mid x \in A \ \& \ x \in B\}$;
- set difference: $A \setminus B := \{x \in A \cup B \mid x \in A \ \& \ x \notin B\}$;
- general intersection: if $\mathcal{A} \neq \emptyset$,

$$\bigcap \mathcal{A} := \{x \in \bigcup \mathcal{A} \mid \forall B.\, (B \in \mathcal{A} \Rightarrow x \in B)\};$$

Set Construction

Cartesian Product

- A, B: sets
- informal definition: $A \times B := \{(x, y) \mid x \in A \ \& \ y \in B\}$;
- formal definition:

$$A \times B := \{w \in 2^{2^{A \cup B}} \mid \exists x. \exists y. (w = (x, y) \ \& \ x \in A \ \& \ y \in B)\};$$

Set Construction

Disjoint Union

- A, B: sets
- $A \uplus B := (\{0\} \times A) \cup (\{1\} \times B)$

Set Construction

Why subset axiom requires a super set?

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- Russell's Paradox: $X := \{x \mid x \notin x\}$;

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Why subset axiom requires a super set?

- Russell's Paradox: $X := \{x \mid x \notin x\}$;
- the paradox:
  - $X \in X \Rightarrow X \notin X$
  - $X \notin X \Rightarrow X \in X$

Set Construction

Why subset axiom requires a super set?

- Russell's Paradox: $X := \{x \mid x \notin x\}$;
- the paradox:
  - $X \in X \Rightarrow X \notin X$
  - $X \notin X \Rightarrow X \in X$
- explanation: $\{x \mid x \notin x\}$ conceptually exists, but is not a set.

Natural Numbers

- $0 := \emptyset$;
- $1 := \emptyset \cup \{\emptyset\}$;
- $n^+ := n \cup \{n\}$ for any natural number n;

Natural Numbers

Inductive Sets

A set A is inductive if $\emptyset \in A$ and for any $a \in A$, $a^+ := a \cup \{a\} \in A$.

Infinity Axiom

There exists an inductive set: $\exists A. [\emptyset \in A \ \& \ (\forall a. a \in A \Rightarrow a^+ \in A)]$.

Natural Numbers

Definition

- A : an inductive set;
- natural numbers:

$$\omega := \mathbb{N} := \{n \in A \mid \forall B.\, (B \text{ is inductive} \Rightarrow n \in B)\}$$

- $\omega = \mathbb{N} = \{0, 1, 2, \dots\}$

Numbers

- the set of integers
- the set of rational numbers
- the set of real numbers
- the set of complex numbers

On Blackboard

Relations and Functions

Relations

Definition

- x, y: sets (objects)
- ordered pairs: $(x, y) := \{\{x\}, \{x, y\}\}$;

A relation $\mathcal{R}$ is a set of ordered pairs.

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Intuition

$(x, y) \in \mathcal{R}$ means x, y are related by $\mathcal{R}$ in order.

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A relation $\mathcal{R}$ is a set of ordered pairs.

Intuition

$(x, y) \in \mathcal{R}$ means x, y are related by $\mathcal{R}$ in order.

Notation

$x\mathcal{R}y : (x, y) \in \mathcal{R}$

Examples

- $\mathcal{R} = \emptyset$;
- $\mathcal{R} = \{(0,1),(0,2),(2,1),(1,2),(4,1)\}$;
- $\mathcal{R} = \mathbb{N} \times \mathbb{N}$;
- $\mathcal{R} = \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid m = 2 \cdot n\}$;
- $\mathcal{R} = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$;

Binary Relations

Definition

- X, Y: sets

A binary relation $\mathcal{R}$ between X and Y is a subset $\mathcal{R} \subseteq X \times Y$ of $X \times Y$.

Binary Relations

Images

- X, Y: sets
- $\mathcal{R} \subseteq X \times Y$: a binary relation
- direct image: for any $A \subseteq X$,

$$\mathcal{R}(A) := \{y \in Y \mid \exists x \in A. x \mathcal{R} y\}$$

- inverse image: for any $B \subseteq Y$,

$$\mathcal{R}^{-1}(B) := \{x \in X \mid \exists y \in B. x \mathcal{R} y\}$$

Binary Relations

Examples

- $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$
  - $\mathcal{R}(\{-1\}) = \{0\}$
  - $\mathcal{R}^{-1}(\{0\}) = \{-1, 1\}$
- $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$
  - $\mathcal{R}(\{1\}) = [1, \infty)$
  - $\mathcal{R}^{-1}(\{1\}) = (-\infty, 1]$

Composition

Definition

- $\mathcal{R}$: a binary relation between X and Y
- $\mathcal{S}$: a binary relation between Y and Z

$\mathcal{S} \circ \mathcal{R}$ is the binary relation between X and Z

$$\mathcal{S} \circ \mathcal{R} := \{(x, z) \in X \times Z \mid \exists y. (x \mathcal{R} y \ \& \ y \mathcal{S} z)\}$$

Composition

- $\mathcal{R} \subseteq X \times X$

Repeated Composition

- $\mathcal{R}^0 := Id_{\mathcal{R}} := \{(x, y) \in X \times X \mid x = y\}$;
- $\mathcal{R}^{n+1} := \mathcal{R} \circ \mathcal{R}^n$

Closures

- $\mathcal{R}^+ := \bigcup_n \mathcal{R}^{n+1}$;
- $\mathcal{R}^* := \bigcup_n \mathcal{R}^n$;

Composition

Closures

- transitive closure: $\mathcal{R}^+ := \bigcup_n \mathcal{R}^{n+1}$;
- reflexive transitive closure: $\mathcal{R}^* := \bigcup_n \mathcal{R}^n$;

Properties

- transitivity: for any $x, y, z \in X$,

$$x\mathcal{R}^+y \ \& \ y\mathcal{R}^+z \Rightarrow x\mathcal{R}^+z$$

- $\mathcal{R}^*$ is in addition reflexive: for any $x \in X$, $x\mathcal{R}^*x$;

Equivalence Relations

Definition

- X: a set
- $\mathcal{R} \subseteq X \times X$: a binary relation

$\mathcal{R}$ is an equivalence relation on X if:

- reflexibility: for any $x \in X$, $x \mathcal{R} x$;
- symmetry: for any $x, y \in X$, $x \mathcal{R} y \Leftrightarrow y \mathcal{R} x$;
- transitivity: for any $x, y, z \in X$, $x \mathcal{R} y \ \& \ y \mathcal{R} z \Rightarrow x \mathcal{R} z$.

Equivalence Relations

Examples

- $\{(x, y) \in X \times X \mid x = y\}$;
- $\{(x, y) \in X \times X \mid f(x) = f(y)\}$ (f is a function on X);
- $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid 7|m - n\}$;

Equivalence Relations

Definition

- X: a set
- $\mathcal{R} \subseteq X \times X$: an equivalence relation on X

For any $a \in X$, define the equivalence class of a by

$$[a]_{\mathcal{R}} := \{x \in X \mid x \mathcal{R} a\}$$

Equivalence Relations

Examples

- $\mathcal{R} = \{(x, y) \in X \times X \mid x = y\}$, $[a]_{\mathcal{R}} = \{a\}$;
- $\{(x, y) \in X \times X \mid f(x) = f(y)\}$, $[a]_{\mathcal{R}} = \{x \mid f(x) = f(a)\}$;
- $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid 7 | m - n\}$, $[6]_{\mathcal{R}} = \{n \mid \exists k \in \mathbb{N}. n = 7 \cdot k + 6\}$;

Partial Orders

Definition

- X: a set
- $\mathcal{R} \subseteq X \times X$: a binary relation

$\mathcal{R}$ is a partial order on X if:

- reflexibility: for any $x \in X$, $x \mathcal{R} x$;
- antisymmetry: for any $x, y \in X$, $x \mathcal{R} y \ \& \ y \mathcal{R} x \Rightarrow x = y$;
- transitivity: for any $x, y, z \in X$, $x \mathcal{R} y \ \& \ y \mathcal{R} z \Rightarrow x \mathcal{R} z$.

Partial Orders

Examples

- $\{(x, y) \in X \times X \mid x = y\}$;
- $\leq$ on $\mathbb{N}, \mathbb{Q}, \mathbb{R}$;
- $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m, n \geq 1, m|n\}$;

Functions

- informal vs. set-theoretic definitions

Functions

Intuition

- X, Y: sets

A function from X to Y is a mapping that assigns to each element in X a unique element in Y.

Functions

Intuition

- A single map is of the form $a \mapsto b$ ($a \in X$, $b \in Y$).
- A function is a collection of such maps.
- It will never happen that there exist two maps $a \mapsto b, a \mapsto c$ such that $b \neq c$.

Functions

Characterization

- a single map $a \mapsto b$: an ordered pair $(a, b) \in X \times Y$
- a collection of maps: a binary relation $F \subseteq X \times Y$
- no $a \mapsto b, a \mapsto c$ satisfying $b \neq c$:

$$\text{for any } (a, b), (a, c) \in F, \text{ we have } b = c$$

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- a single map $a \mapsto b$: an ordered pair $(a, b) \in X \times Y$
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$$\text{for any } (a, b), (a, c) \in F, \text{ we have } b = c$$

Example

- $F(x) = x^2, \ x \in \mathbb{R}$
- $F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$

Functions

Set-Theoretic Definition

- X, Y: sets

A partial function F from X to Y is a binary relation $F \subseteq X \times Y$ such that

$$\forall x \in X. \forall y, y' \in Y. \left[xFy \ \& \ xFy' \Rightarrow y = y' \right]$$

Functions

Set-Theoretic Definition

- X, Y: sets

A partial function F from X to Y is a binary relation $F \subseteq X \times Y$ such that

$$\forall x \in X. \forall y, y' \in Y. \left[xFy \ \& \ xFy' \Rightarrow y = y' \right]$$

Notation

- A partial function F from X to Y is stressed by $F : X \rightharpoonup Y$.
- $F(x)$ is define as the unique y such that xFy if such y exists.

Functions

Set-Theoretic Definition

- X, Y: sets

A (total) function F from X to Y is a partial function from X to Y such that for any $x \in X$ there exists $y \in Y$ such that xFy.

Functions

Set-Theoretic Definition

- X, Y: sets

A (total) function F from X to Y is a partial function from X to Y such that for any $x \in X$ there exists $y \in Y$ such that xFy.

Notation

- For each $x \in X$, $F(x)$ is the unique element such that $(x, F(x)) \in F$.
- A function F from X to Y is stressed by $F : X \to Y$.

Functions

- range: $F(X) = \{y \in Y \mid \exists x.y = F(x)\}$
- domain: $F^{-1}(Y) = \{x \in X \mid \exists y.y = F(x)\} = X$

Functions

Examples

- $F : \mathbb{R} \rightharpoonup \mathbb{R}$, $F(x) = \frac{1}{x}$:

$$F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \cdot y = 1\} \ ;$$

- $F : \mathbb{R} \to \mathbb{R}$, $F(x) = \sin x$:

$$F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = \sin x\};$$

- $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F(x, y) = x + y$:

$$F = \{((x, y), z) \in (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \mid z = x + y\};$$

Functions

λ-Notation

- X, Y: sets
- $f : X \to Y$: a function
- e: an expression representing f (e.g., $e = x + 1$ and $f(x) = x + 1$)

Then we denote also by $\lambda x \in X.e$ the function f.

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- X, Y: sets
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Then we denote also by $\lambda x \in X.e$ the function f.

Example

- $\lambda x \in \omega.(x + 1)$: the function $f(x) = x + 1$
- $\lambda x \in \mathbb{R}. \sin x$: the function $f(x) = \sin x$

Composition (Recall)

Definition

- $\mathcal{R}$: a binary relation between X and Y
- $\mathcal{S}$: a binary relation between Y and Z

$\mathcal{S} \circ \mathcal{R}$ is the binary relation between X and Z defined by

$$\mathcal{S} \circ \mathcal{R} := \{(x, z) \in X \times Z \mid \exists y. (x \mathcal{R} y \ \& \ y \mathcal{S} z)\}$$

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Example

- $F : X \to Y$
- $G : Y \to Z$

$G \circ F : X \to Z$ satisfies that $(G \circ F)(x) = G(F(x))$

Cantor's Diagonal Argument

Inverse

- X, Y: sets

A function $F : X \to Y$ has an inverse $G : Y \to X$ if

- $G(F(x)) = x$ for all $x \in X$, and
- $F(G(y)) = y$ for all $y \in Y$.

If there exists a function $F : X \to Y$ with its inverse $G : Y \to X$, then X, Y are in 1-1 correspondence.

Cantor's Diagonal Argument

Theorem

- X: a set

X and 2^X are not in 1-1 correspondence.

Cantor's Diagonal Argument

Theorem

X and 2^X are not in 1-1 correspondence.

Proof (by Contradiction)

$\theta : X \to 2^X$ with an inverse

	$\theta(x_0)$	$\theta(x_1)$	$\theta(x_2)$	$\dots$	$\theta(x_j)$	$\dots$
x_0	0	1	1	$\dots$	1	$\dots$
x_1	1	1	1	$\dots$	0	$\dots$
x_2	0	0	1	$\dots$	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
x_i	0	1	0	$\dots$	1	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Cantor's Diagonal Argument

Theorem

X and 2^X are not in 1-1 correspondence.

Proof

$\theta : X \to 2^X$ with an inverse

	$\ldots$	$Y = \theta(y)$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$
y	$\ldots$	$y \in \theta(y)?$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$

Cantor's Diagonal Argument

Theorem

X and 2^X are not in 1-1 correspondence.

Proof

- Suppose that there is a function $\theta : X \to 2^X$ with an inverse.
- Define the set

$$Y := \{x \in X \mid x \notin \theta(x)\} \in 2^X \ .$$

- Let y be the unique element in X such that $\theta(y) = Y$.

Cantor's Diagonal Argument

Theorem

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Proof

- Suppose that there is a function $\theta : X \to 2^X$ with an inverse.
- Define the set

$$Y := \{x \in X \mid x \notin \theta(x)\} \in 2^X \ .$$

- Let y be the unique element in X such that $\theta(y) = Y$.
- $y \in Y \Rightarrow y \notin \theta(y) (= Y)$
- $y \notin Y \Rightarrow y \in \theta(y) (= Y)$

Summary

Basic Set Theory

- a deeper understanding of sets
- axioms for set reasoning and construction
- set-theoretic definitions for relations and functions
- rigorous reasoning with relations and functions

Questions?