

# Lecture 1. Introduction

Mathematical optimization problem

minimize  $f(x)$

subject to  $x \in \Omega$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  objective function 目标函数

$x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$  optimization variables.

$\Omega \subseteq \mathbb{R}^n$  feasible set / constraint set

-  $x \in \Omega$  feasible 可行 infeasible o.w.

-  $\Omega$  specified by constraint functions  $g_1, \dots, g_m$ .

$\min_x f(x)$

s. t.  $g_i(x) \leq 0, \quad i=1, \dots, m.$

$x^*$ : optimal solution.  $f(x)$  achieves optimal.

$x^* = \operatorname{argmin} f(x).$

Remark: maximizing  $f(x)$  is equivalent to minimizing  $-f(x)$

why linear and convex?

In general optimization problems are very difficult to solve.

Knapsack problem. 背包问题

$n$  types of knapsacks.

$i^{\text{th}}$  type carry  $a_i$  pencils and  $b_i$  books. costs  $c_i$

$A$  pencils and  $B$  books in total.

Goal: spend least money.

$$\min \sum_i c_i x_i$$

$$\text{s.t. } \sum_i a_i x_i \geq A$$

$$\sum_i b_i x_i \geq B.$$

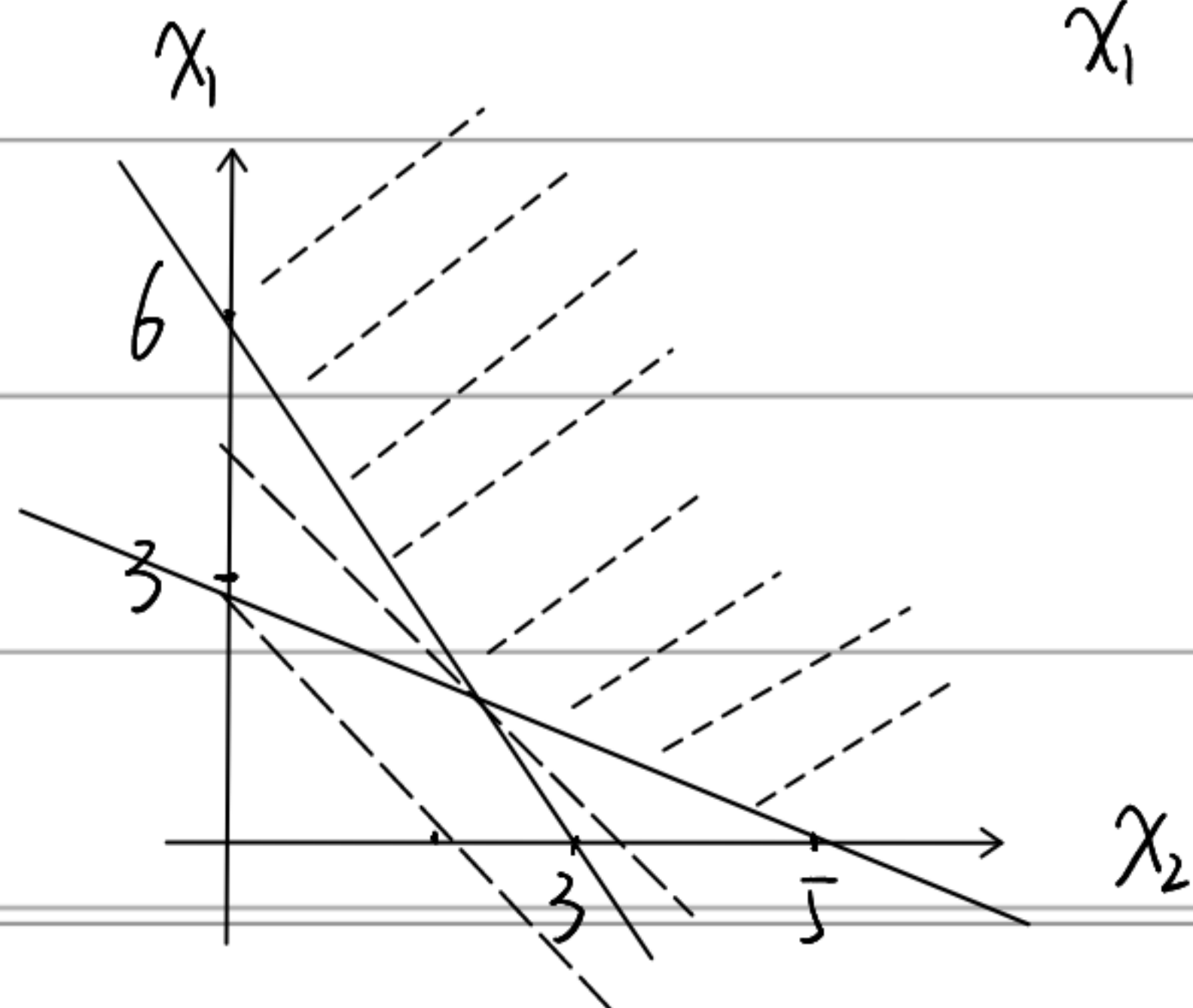
How to solve it?

if only 2 types.

$$\min 10x_1 + 15x_2.$$

$$\text{s.t. } 5x_1 + 3x_2 \geq 15$$

$$x_1 + 2x_2 \geq 6$$



more types?

simplex algorithm.

primal dual

# Data fitting.

T	V
30	1.011
40	1.019
50	1.032
60	1.041
...	...

$$\begin{array}{c} V \\ \downarrow \\ y = kx + b. \end{array} \quad \begin{array}{c} T \\ \downarrow \end{array}$$

what are the two coefficient  
k and b ?

## Least squares method.

given  $n$  measurements.

$(x_1, y_1), \dots, (x_n, y_n).$

assume  $i^{\text{th}}$  error  
denoted by  $\epsilon_i$ .

least squares criterion

minimize

$$\sum \epsilon_i = \sum (y_i - kx_i - b)^2$$

## Geometric explanation : projection

$$\begin{cases} 30k + b = 1.011 \\ 40k + b = 1.019 \\ 50k + b = 1.032 \end{cases}$$

$$k \begin{bmatrix} 30 \\ 40 \\ 50 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = ?$$

projection of  $\begin{bmatrix} 1.011 \\ 1.019 \\ 1.032 \end{bmatrix}$  onto the subspace spanned by

minimize  $\left\| \begin{bmatrix} 30 & 1 \\ 40 & 1 \\ 50 & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} - \begin{bmatrix} 1.011 \\ 1.019 \\ 1.032 \end{bmatrix} \right\|_2$   $\begin{bmatrix} 30 \\ 40 \\ 50 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Norm. inner product. distance to hyperplane

not necessary  $\mathbb{R}^n$   
Fourier

IP: An inner product  $\langle \cdot, \cdot \rangle$  is a function  $S \times S \rightarrow \mathbb{R}$  s.t.

1. nonnegative  $\langle x, x \rangle \geq 0$   $= 0$  iff  $x = 0$

2. symmetric  $\langle x, y \rangle = \langle y, x \rangle$

3. linearity  $\langle sx, y \rangle = s \langle x, y \rangle$  (homogeneity)

$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad (\text{additivity})$$

if  $\langle x, y \rangle = 0$  then  $x$  and  $y$  are called orthogonal.  $\perp$  交.

Euclidean inner product space:  $\langle x, y \rangle = x^T y = \sum x_i y_i$

$\|\cdot\|$ .

Norm: a norm is a function  $\mathbb{R}^n \rightarrow \mathbb{R}$  s.t.

1. nonnegative  $\|x\| \geq 0$   $= 0$  iff  $x = 0$ .

2. positive homogeneity  $\|a x\| = |a| \|x\|$

3. triangle inequality  $\|x+y\| \leq \|x\| + \|y\|$ .

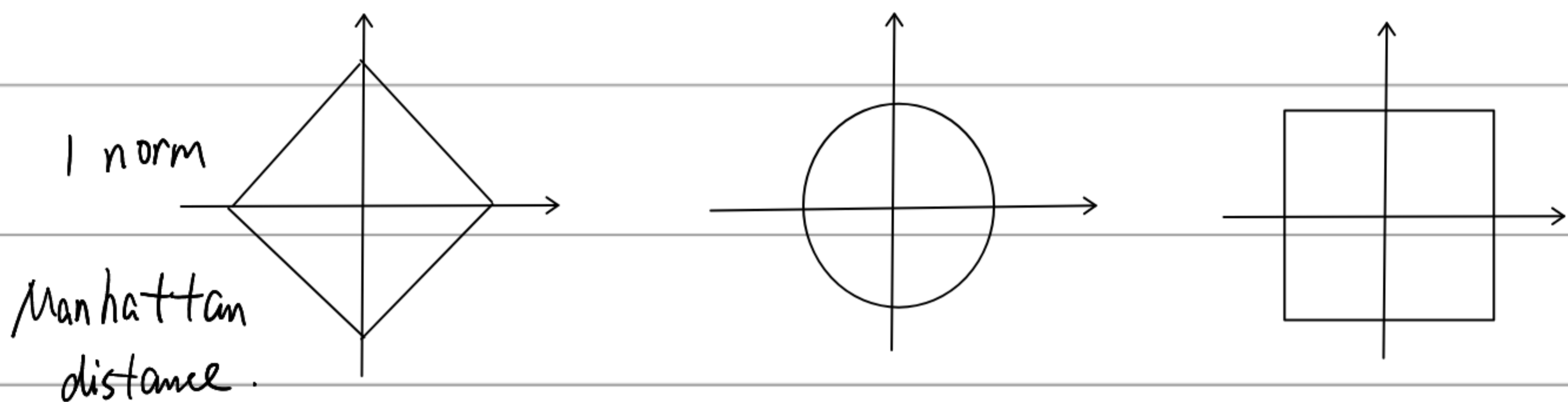
$L^p$ -norm, or  $p$ -norm for real  $p \geq 1$ .

$$\|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$$

in particular. 1-norm:  $\|x\|_1 = \sum_i |x_i|$

(Euclidean norm) 2:  $\|x\| \stackrel{\Delta}{=} \|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{\sum_i x_i^2}$   
default

$\infty$ -norm:  $\|x\|_{\infty} = \max \{ |x_1|, |x_2|, \dots, |x_n| \}$ .



Cauchy - Schwarz inequality

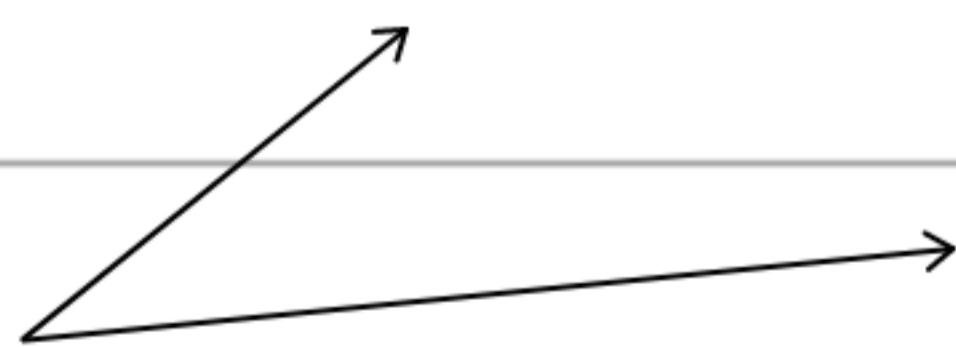
$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle \quad \text{or.}$$

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

$n$ -dimensional Euclidean space. ( $\mathbb{R}^n$ )

$$\left( \sum_i u_i v_i \right)^2 \leq \left( \sum_i u_i^2 \right) \left( \sum_i v_i^2 \right).$$

Geometric explanation: projection.



Distance to hyperplane.

hyperplane  $P$ :  $w^T x + b = 0$

$w \perp P$ .

orthogonal projection

$(x - x') \perp P \quad w^T x' + b = 0.$

