Lecture 10. Two phase Simplex method; Duality of LP

Simplex method: start from origin. move and shift.

If origin is not feasible? Suppose $x=d$ is feasible.

let $y_i = d_i - x_i$. rewrite the LP so that $y=0$ is feasible.

y_i may be less than $0 \rightarrow y_i = y_i^+ - y_i^- = d_i - x_i$. origin is a vertex

How to find a feasible solution?

Consider a LP: $\min c^T x$. s.t. $Ax = b$, $x \geq 0$.

add slack variables $s_1, \dots, s_m$. s.t. $A_i x + s_i = b_i$. $\forall i$.

a trivial solution $x_1 = x_2 = \cdots = x_n = 0$, $s_i = b_i$. $\forall i$.

consider the LP: $\min s_1 + s_2 + \cdots + s_m$. $A_i x + s_i = b_i$. $\forall i$.

if optimal $=0$. find a feasible sol. of the initial LP. o.w. infeasible.

Two more questions: correctness and running time.

Correctness: simplex method finds a local optimum.

if halts. origin is the local minimum. every coefficient is ≥ 0.

for initial LP. $x_1, \dots, x_k$ are neighbours of x^*. $\forall y \in P$.

$y - x^*$ is a conic combination of $(x_1 - x^*)$, $(x_2 - x^*)$, $\dots$, $(x_k - x^*)$.

Running time: avoid singular bad instance by perturbation (smooth analysis).

Consider a LP $\min\ 3x_1 + 2x_2$ s.t. $x_1 + x_2 \le 5,\ x_1 \le 3,\ x_2 \le 4$.

Trivially $x_1 = x_2 = 0$ is the optimal solution since $x_1, x_2 \ge 0$

How about $\min\ -3x_1 - 2x_2$? (equivalent to $\max\ 3x_1 + 2x_2$).

$3x_1 + 2x_2 \le 13$ since $3x_1 + 2x_2 \le 2(x_1 + x_2) + x_1 \le 13$.

Given $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ $b \in \mathbb{R}^m$. assign y_i to each constraint

$$\max\ c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

$$\begin{array}{llr} \text{s.t.} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1. & y_1 \\ & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2. & y_2 \\ & \vdots & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m & y_m. \end{array}$$

if $y_j \ge 0$. and $\sum y_i a_{ij} \ge c_j \ \forall j$. $\sum y_i b_i$ is an upper bound.

$$\Rightarrow \max\ c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \le \min\ y_1 b_1 + y_2 b_2 + \cdots + y_m b_m.$$

Construct the LP: $\min\ y^T b$. s.t. $y^T A \ge c^T$ $y \ge 0$.

is the dual of primal LP $\max\ c^T x$ s.t. $Ax \le b$. $x \ge 0$.

Proposition. The dual of the dual is the primal.

Proof. Rewrite dual as $\max\ -b^T y$. s.t. $-A^T y \le -c,\ y \ge 0$.

the dual of the dual is $\min\ -w^T c$. s.t. $-w^T A \ge -b^T$. $w \ge 0$. $\square$

Weak duality theorem $\longleftarrow$ - - - - - $\longrightarrow$ primal feasible dual feasible.

If x is feasible for primal. y is feasible for dual. then $c^Tx \le y^Tb$

Proof. $c^Tx \le y^TAx \le y^Tb$. $(x \geq 0,\ y \geq 0)$. $\square$

Corollary. If primal has optimal z. dual has optimal w. $z \le w$.

(here for convention. let $\max \phi = -\infty$ and $\min \phi = \infty$).

Strong duality theorem. $\longleftarrow \bullet \longrightarrow$ primal feasible dual feasible.

If primal has finite optimal x^*. so is dual. and $c^Tx^* = y^{*T}b$.

primal \ dual	unbounded	infeasible	feasible
unbounded	X	✓	X
infeasible	✓	✓	X
feasible.	X	X	✓

weak duality: unbounded $\Rightarrow$ infeasible. others are impossible.

strong duality: feasible $\Rightarrow$ feasible. but both infeasible?

Primal: $\max\ 2x_1 - x_2$. s.t. $x_1 - x_2 \le 1$. $-x_1 + x_2 \le -2$. $x_1, x_2 \geq 0$.

dual: $\min\ y_1 - 2y_2$ s.t. $y_1 - y_2 \geq 2$. $-y_1 + y_2 \geq -1$. $y_1, y_2 \geq 0$

Proof of strong duality: review of Farkas' lemma.

let $A \in \mathbb{R}^{m\times n}$. $b \in \mathbb{R}^m$. then exactly one of the followings is true.

① $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$. $x \geq 0$. ② $\exists y \in \mathbb{R}^m$ s.t. $A^Ty \geq 0$. $b^Ty < 0$.