Lecture 11. Applications of duality; Descent method

Zero-sum games: rock-scissors-paper game.

	R	S	P
R	0	1	-1
S	-1	0	1
P	1	-1	0

$G \in \mathbb{R}^{n\times n}$: payoff matrix.

x, y : strategy distribution over $\{R, S, P\}$.

expected payoff : $\mathbb{E}[\text{payoff}] = \sum_{i,j} G_{ij} x_i y_j$.

$x = y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. $\mathbb{E}[\text{payoff}] = 0$.

$x = (0, 0, 1)$. $y = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. $\mathbb{E}[\text{payoff}] = \frac{1}{4}$.

$x = (0, 0, 1)$. $y = (0, 1, 0)$. $\mathbb{E}[\text{payoff}] = -1$.

Player X: for fixed x. player Y's best strategy is to minimize $\sum_{i,j} G_{ij} x_i y_j$. so X's strategy is to max min.

Player Y: for fixed y. player X's best strategy is to maximize $\sum_{i,j} G_{ij} x_i y_j$. so Y's strategy is to min max.

Min max Theorem: $\max_x \min_y x^T G y = \min_y \max_x x^T G y$.

Example. $G = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$.

if X first. with strategy (x_1, x_2). payoff of Y is $\begin{pmatrix} 3x_1 - 2x_2 \\ -x_1 + x_2 \end{pmatrix}$

goal of X : $\max_{x_1 + x_2 = 1} \min \{3x_1 - 2x_2, -x_1 + 2x_2\}$

$\max z$. s.t. $3x_1 - 2x_2 \geq z$. $-x_1 + x_2 \geq z$. $x_1 + x_2 = 1$. $x_1, x_2 \geq 0$

if Y first with strategy (y_1, y_2). payoff of X is $\begin{pmatrix} 3y_1 - y_2 \\ -2y_1 + y_2 \end{pmatrix}$

goal of Y: $\min_{y_1 + y_2 = 1} \max \{3y_1 - y_2, -2y_1 + y_2\}$.

$\min w$. s.t. $3y_1 - y_2 \leq w$. $-2y_1 + y_2 \leq w$. $y_1 + y_2 = 1$. $y_1 y_2 \geq 0$.

min max is the dual of max min. so equality by strong duality.

Matching. vertex cover and fractional matching / vertex cover.

matching: a subset of edges. $S \subseteq E$. s.t. $\forall v \in V$.

at most one of incident edges is in S.

vertex cover: a subset of vertices. $T \subseteq V$. s.t. $\forall e \in E$.

at least one of incident vertices is in T.

max matching: $\max \sum x_e$. s.t. $\sum_{e \sim v} x_e \leq 1$. $\forall v \in V$

min vertex cover: $\min \sum y_v$ s.t. $\sum_{v \sim e} y_v \geq 1$. $\forall e \in E$.

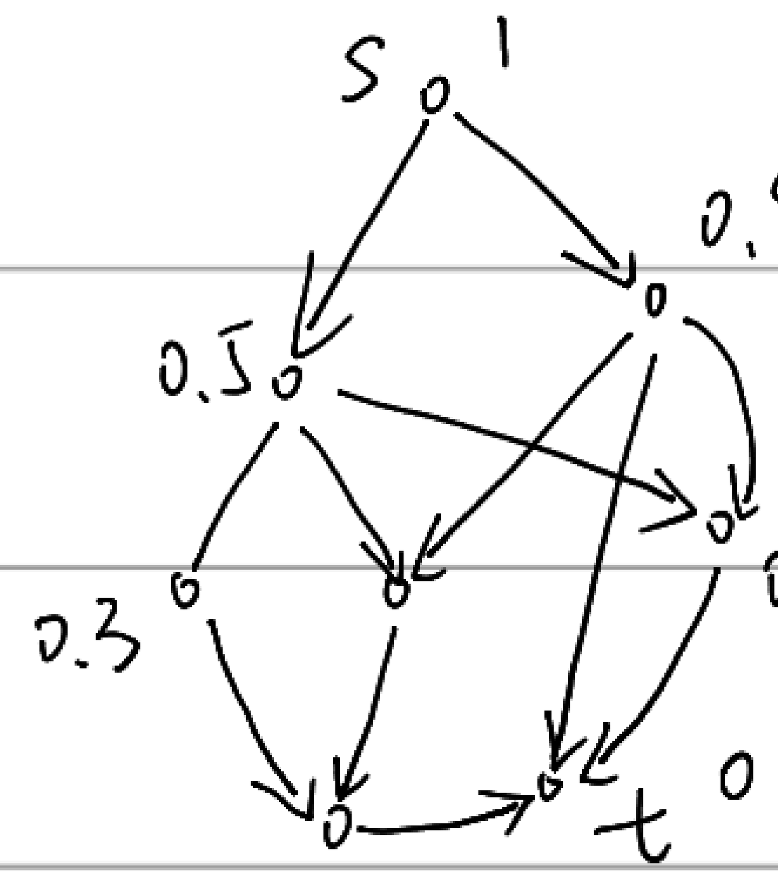
if $x_e, y_v \in \{0, 1\}$. done. otherwise define fractional.

optimal solution at vertex of LP $\Rightarrow$ no need to consider irrational.

Thm: max fractional matching = min fractional vertex cover.

in particular, in bipartite graph. max matching = min vertex cover.

Max flow and min cut: consider DAG for convenience.



DAG having a source and a terminal.

each edge has a capacity c_e.

for each vertex except s.t. flow-in = flow-out.

max flow: $\max \sum_u x_{su}$ s.t. $0 \le x_{uv} \le c_{uv}$. $\sum_u x_{uv} - \sum_w x_{vw} = 0$. $\forall v$.

cut: select a subset S of vertices. s.t. $s \in S$. $t \notin S$.

$\text{cut}(S, \bar{S}) = \{(u,v) : u \in S, v \in \bar{S}\}$. $\Longleftrightarrow$ a subset T of edges E.

s.t. $\forall$ path P from s to t. $T \cap P \neq \phi$.

min cut: $\text{minimize}_S \sum_{e \in \text{cut}(S)} c_e$.

dual of max flow: $\min \sum_{u,v} c_{uv} w_{uv}$ s.t. $\begin{cases} w_{su} + y_u \ge 1 \\ w_{vt} - y_v \ge 0. \\ w_{uv} + y_v - y_u \ge 0. \end{cases}$

let $y_s = 1$. $y_t = 0$. then $w_{uv} \ge y_u - y_v$.

if $y_v \in \{0,1\}$. $w_{uv} \in \{0,1\}$ for any u.v. it is min cut.

Select a cut randomly (the probabilistic method).

pick $\rho \in (0,1)$ uniformly at random. let $S = \{v : y_v > \rho\}$.

$\mathbb{E}\left[\sum_{(u,v)\in \text{cut}(S)} c_{uv}\right] = \mathbb{E}\left[\sum_{(u,v)} c_{uv} \cdot \mathbf{1}[(u,v) \in \text{cut}(S)]\right]$

① $\mathbf{1}[P]$ is an indicator random variable that is 1 if P is true and 0 otherwise.

$= \sum_{(u,v)} c_{uv} \cdot \mathbb{E}[\mathbf{1}[(u,v) \in \text{cut}(S)]]$

② $\mathbb{E}[X] \le y \Rightarrow \exists \omega \in \Omega. X(\omega) \le Y$.

$= \sum_{(u,v)} c_{uv} \cdot \Pr((u,v) \in \text{cut}(S)) = \sum_{(u,v)} c_{uv} \cdot (y_u - y_v) \le \sum c_{uv} w_{uv}$. $\square$