Lecture 4. Geometry: affine and convex sets.

Heine-Borel Theorem: $\mathbb{R}^n$. bounded closed $\Leftrightarrow$ compact.

line: $z = x + \theta(y-x) = \theta y + (1-\theta)x$. $\theta \in \mathbb{R}$.

affine set: $\forall x, y \in S$. $\forall \theta$. $z \in S$.

affine combination given $x_1, \cdots, x_n \in S$. $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$. s.t. $\theta_1 + \theta_2 + \cdots + \theta_n = 1$.

Example: solution set of linear equations

$\{x : Ax = b\}$.

why affine? $S - x_0$ linear space. so linear + offset.

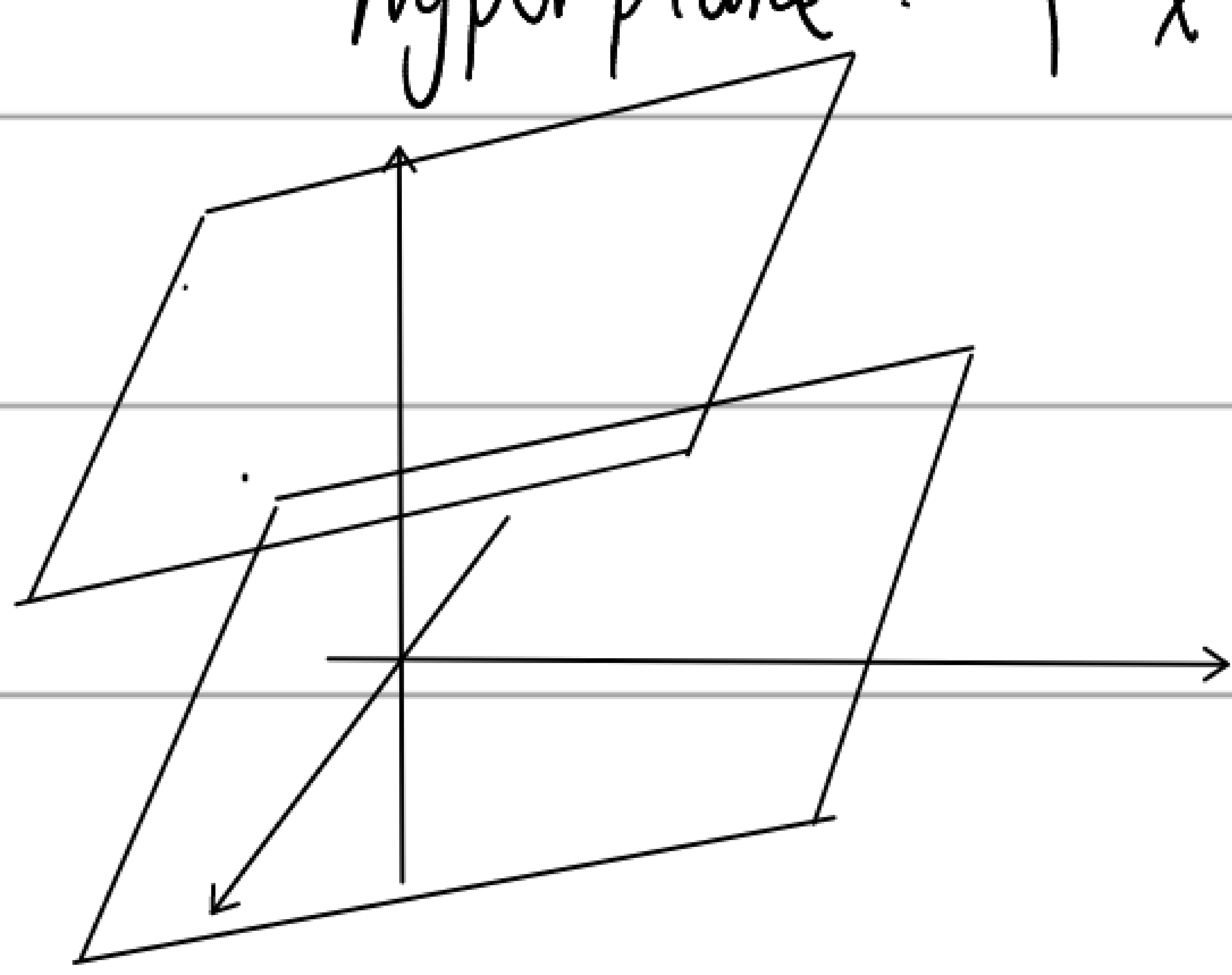
conversely every affine set can be expressed as solution set of linear equations.

$x+b$, $y+b \in S$.

$\alpha x + \beta y + b = \alpha(x+b) + \beta(y+b) + (1-\alpha-\beta)b \in S$.

in particular. If $A \in \mathbb{R}^{1\times n}$

hyperplane: $\{x : w^T x = b\}$. $w \neq 0$.



affine set is an intersection of finite hyperplane

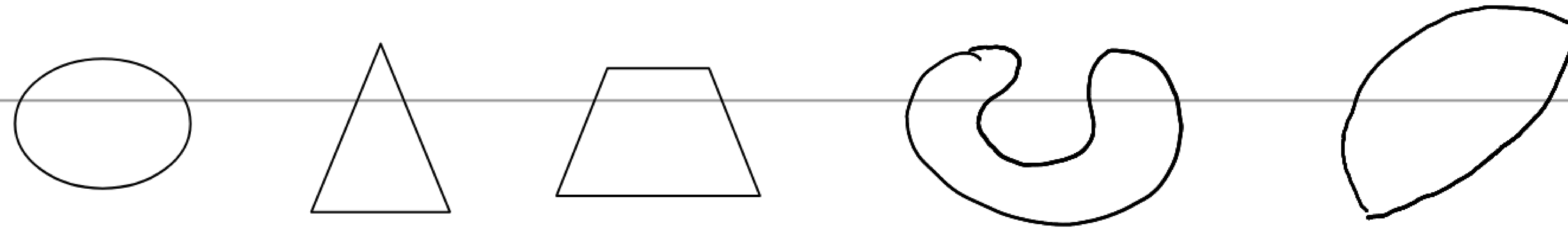
if $0 \in$ hyperplane. a $(n-1)$-dim subspace.

If $A \neq 0$ in affine set. $\Rightarrow$ intersection of finite hyperplane.

segment: $z = x + \theta(y-x) = \theta y + (1-\theta)x$. $\theta \in [0,1]$

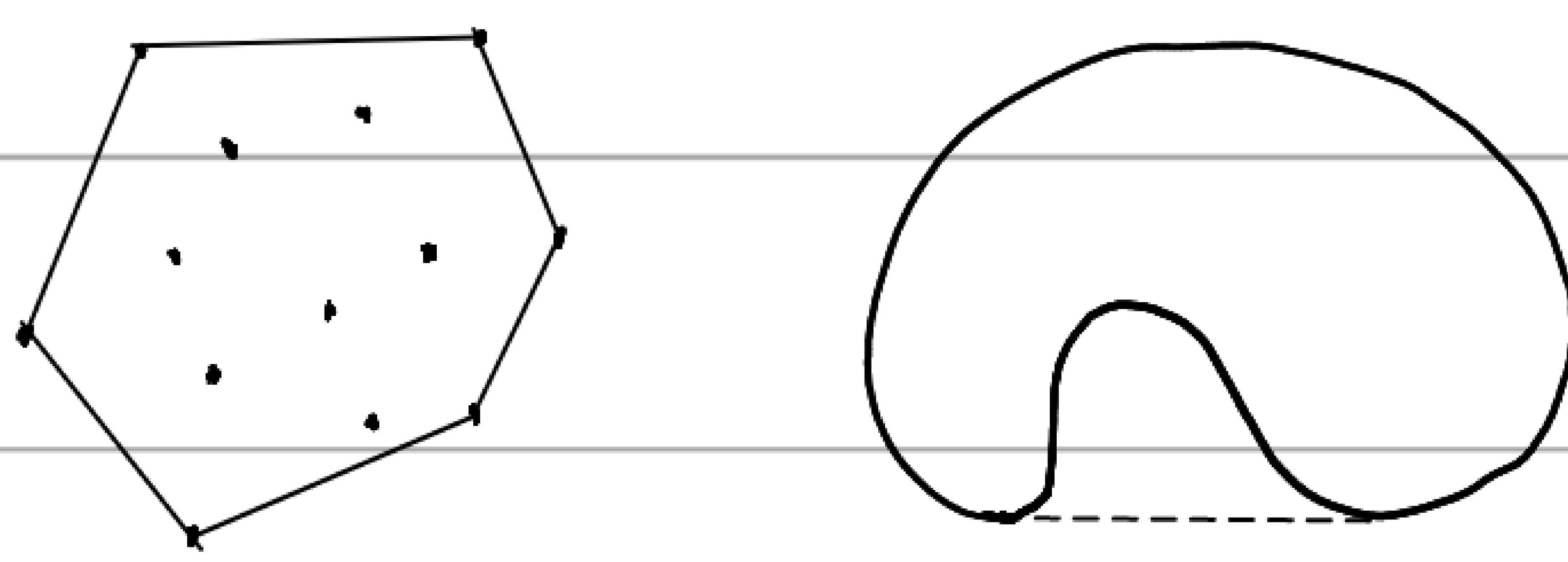
convex set: $\forall x, y \in S$. $\forall \theta \in [0,1]$. $z \in S$.

凹 is not convex.



convex combination: $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$. $\theta_i \geq 0$. $\Sigma \theta_i = 1$.

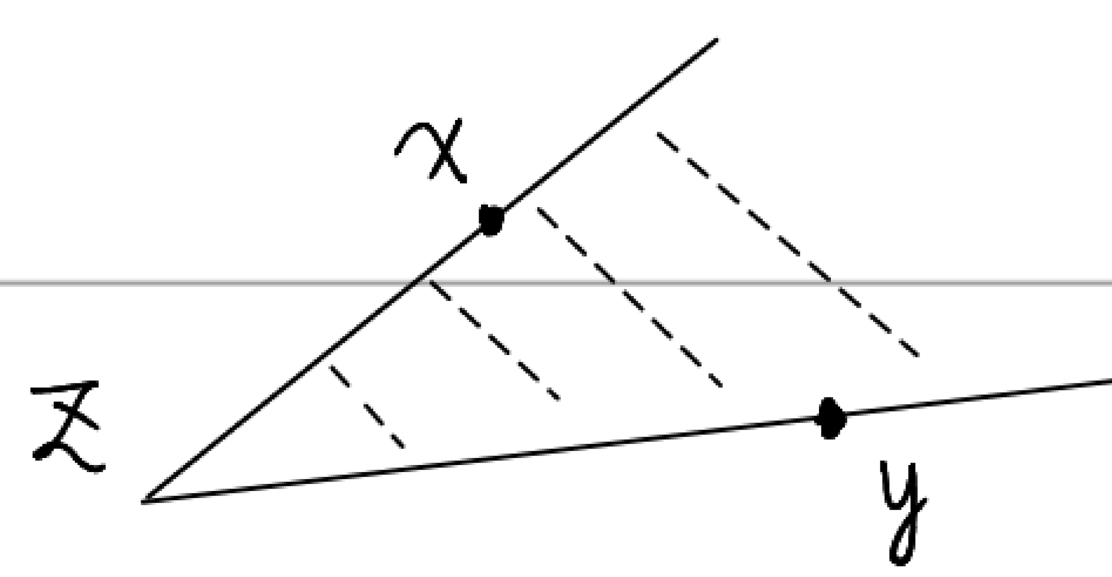
S is convex iff S contains every convex combination of its points.



convex hull: set of all convex combination of points in S.

$\{ \Sigma \theta_i x_i : \theta_i \geq 0. \Sigma \theta_i = 1. x_i \in S \}$.

conic combination. $z = \theta_1 x + \theta_2 y$. $\theta_1 . \theta_2 \geq 0$.



convex cone: set that contains all conic combination of points in S.

Some examples of convex sets:

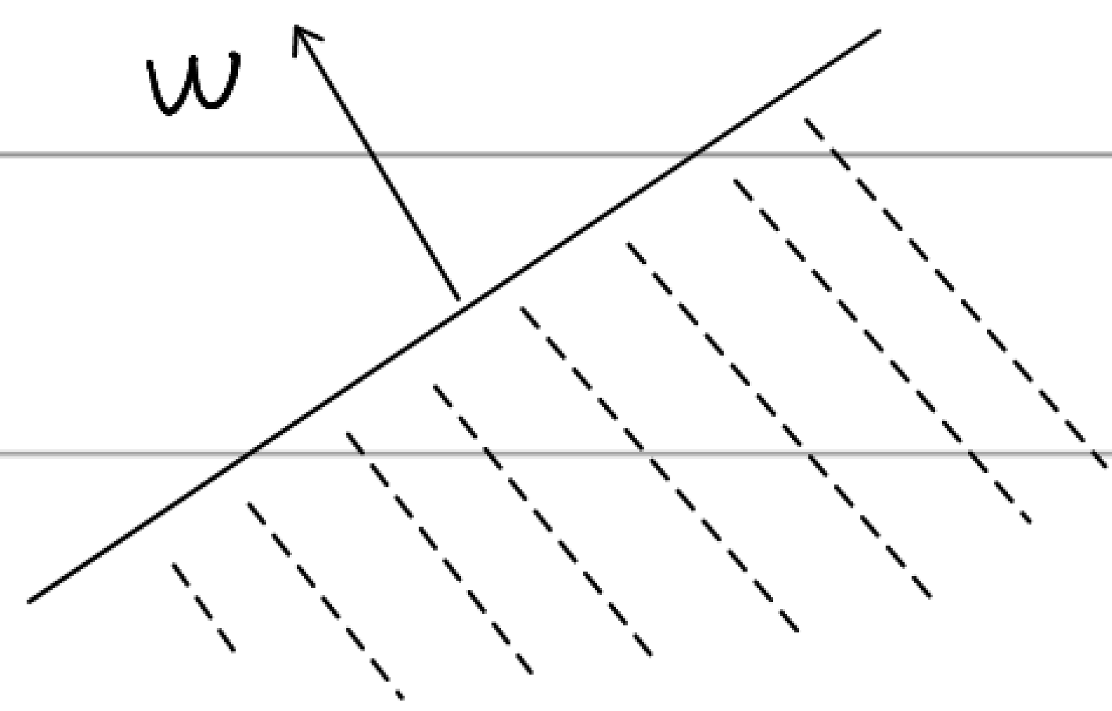
$\mathbb{R}^n$. affine sets. hyperplanes.

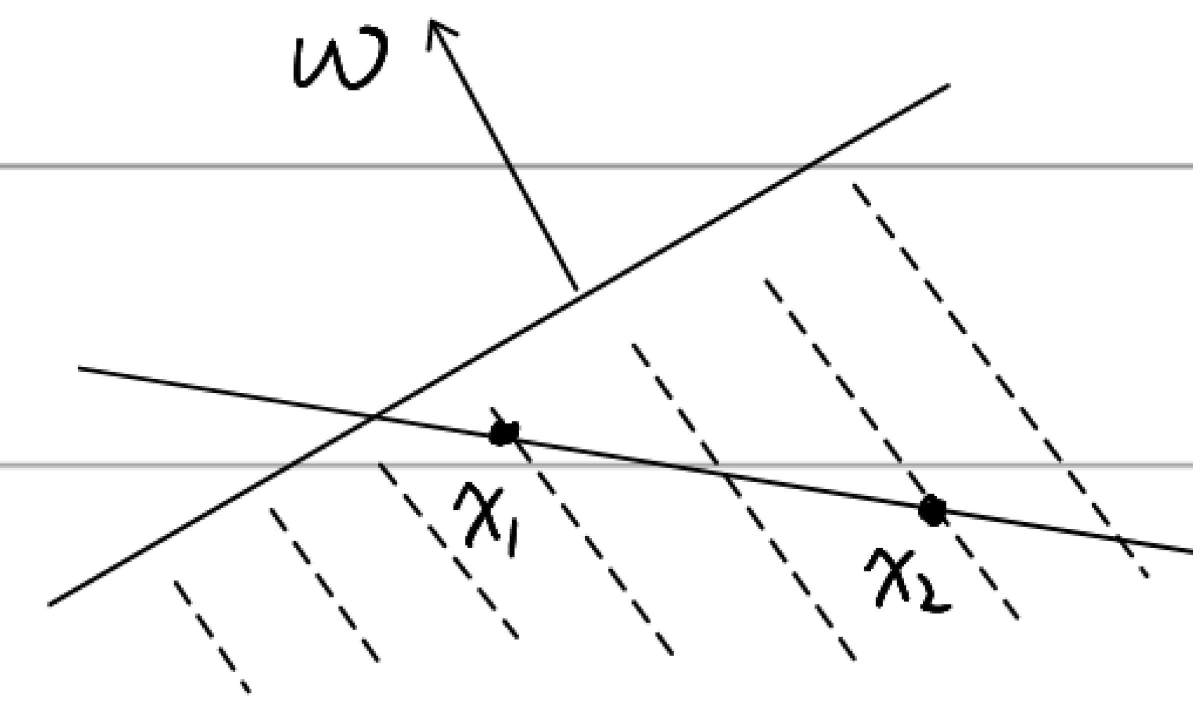
half spaces: a hyperplane divide $\mathbb{R}^n$ into 2 halfspaces.

$\{x : w^T x = b\}$. $\{x : w^T x < b\}$. $\{x : w^T x > b\}$

closed halfspace: $\{x : w^T x \leq b\}$ $w \neq 0$.

halfspaces are convex, but not affine.





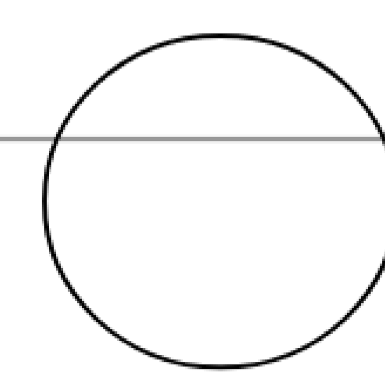
open halfspace. $\{x : w^T x < b\}$ interior points of closed.

$\bar\theta \triangleq 1-\theta$. $\forall x, y \in S = \{x : w^T x \le b\}$.

$$w^T(\theta x + \bar\theta y) = \theta w^T x + \bar\theta w^T y \le \theta \cdot b + \bar\theta \cdot b = b.$$

Euclidean balls and ellipsoids.

Euclidean ball:



$\{x : \|x - x_0\|_2 \le r\}$.

$\iff \{x : x = x_0 + r d,\ \|d\|_2 \le 1\}$.

triangle inequality.

$$\|\theta x + \bar\theta y - x_0\|_2 = \|\theta(x-x_0) + \bar\theta(y-x_0)\|_2 \le \cdots$$

norm ball: $\{x : \|x - x_0\| \le r\}$.

Ellipsoid: $\{x : \frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} \le 1\}$. convex.

Proof: $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$. $E = \{\Lambda u : \|u\|_2 \le 1\}$

suppose $x_i = \Lambda u_i$. $\theta x_1 + \bar\theta x_2 = \Lambda(\theta u_1 + \bar\theta u_2) = \Lambda u$

for some $\|u\|_2 \le 1$.

$E = \{x : \|\Lambda^{-1} x\|_2 \le 1\} = \{x : x^T \Lambda^{-2} x \le 1\}$

in general. $E = \{x_0 + \Lambda u : \|u\|_2 \le 1\} = \{x : (x-x_0)^T \Lambda^{-2} (x-x_0) \le 1\}$.
shift

with rotation. $A = Q \Lambda Q^T$. Q is orthogonal. $A \succ 0$ positive definite.

$E = \{x_0 + Au : \|u\|_2 \le 1\} = \{x : (x-x_0)^T A^{-2} (x-x_0) \le 1\}$.

Proposition: the image of a convex set under an affine function is also convex.

proof: $f(x) = Ax + b$ is affine.

$C \subseteq \mathbb{R}^n$ is convex. give x_1, x_2. and $y_i = f(x_i)$

goal: $\forall \theta \in [0,1]$. $\theta y_1 + \bar{\theta} y_2 \in f(C)$.

$\in C$.

$\theta y_1 + \bar{\theta} y_2 = (\theta A x_1 + \theta b) + (\bar{\theta} A x_2 + \bar{\theta} b) = A(\theta x_1 + \bar{\theta} x_2) + b$.

The inverse image of a convex set is also convex.

A nongeometric example: positive semidefinite matrices.

The set of positive semidefinite matrices is convex.

$S_+^n \triangleq \{A \in \mathbb{R}^{n\times n} : A \succeq 0\}$. S_{++}^n positive definite.

proof: 1. symmetric. A. B. symmetric $\Rightarrow \theta A$. $\bar{\theta} B$. symmetric.

2. $x^T(\theta A + \bar{\theta} B)x \geq 0$

$\quad \| $

$\theta x^T A x + \bar{\theta} x^T B x \geq 0$

Intersection: $\{C_i : i \in I\}$ a family of convex sets. $\bigcap_i C_i$ convex.