

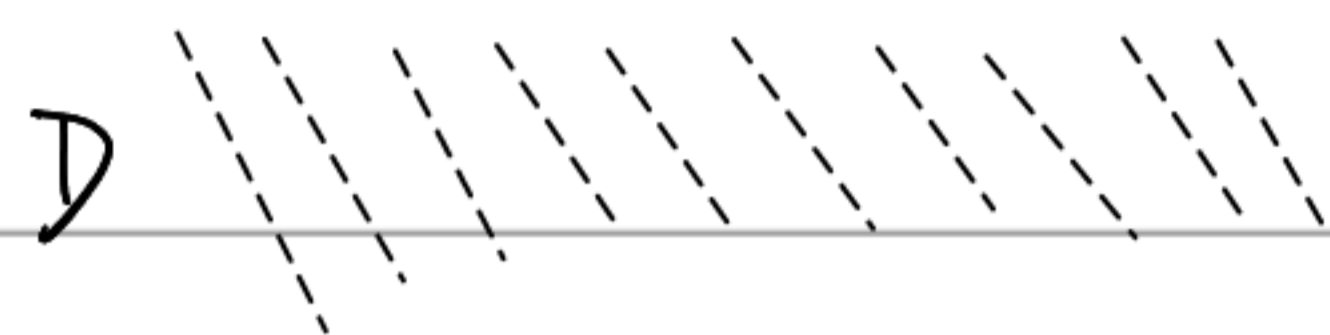
# Lecture 5. Geometry: separating and supporting hyperplane.

Proof of separating hyperplane theorem.

Strict separating theorem.



$C, D$  are two closed convex



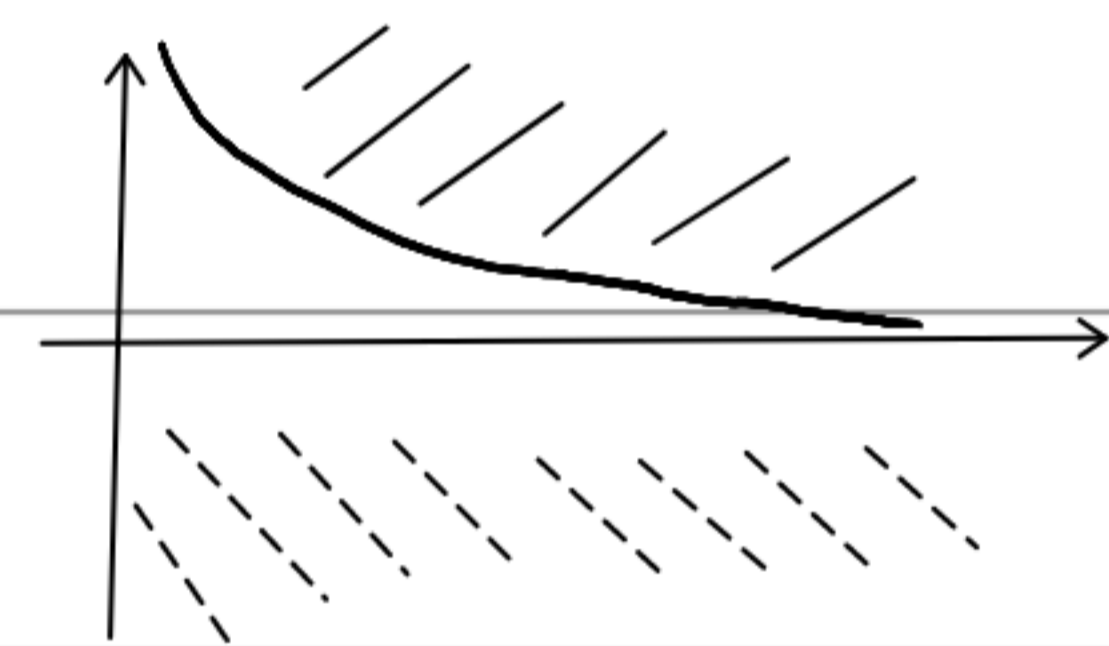
sets and at least one of them is bounded.

then  $\exists$  strict separating hyperplane. i.e.

$$\exists w, b \text{ s.t. } w^T x < b < w^T y \quad \forall x \in C, y \in D.$$

counterexample.

two closed but unbounded set?



Proof of strict separating hyperplane theorem.

First, find  $u \in C, v \in D$  s.t.  $\text{dist}(u, v) = \text{dist}(C, D)$ .

$$\text{where } \text{dist}(C, D) = \inf_{x \in C, y \in D} \|x - y\| \quad \text{why exists?}$$

w.l.o.g.  $C$  bounded. given  $u \in C, v \in D$ .

$D \cap \{x : \|u - x\| \leq \|u - v\|\}$  convex and compact.

$\Rightarrow \exists \bar{v} \in D$   $\text{dist}(u, \bar{v}) = \text{dist}(u, D)$  — continuous function

$C$  compact  $\Rightarrow \exists \underset{u}{\text{argmin}} \text{dist}(u, D) \triangleq \bar{u}$ .

$\bar{u} \in C, \bar{v} \in D$  s.t.  $\|\bar{u} - \bar{v}\| = \text{dist}(C, D)$ .

let  $w = \bar{v} - \bar{u}, b = \frac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2}$ .

our separating hyperplane is  $f(x) = w^T x + b$ .

we claim that  $f(x) > 0, \forall x \in D$  and  $f(x) < 0, \forall x \in C$ .

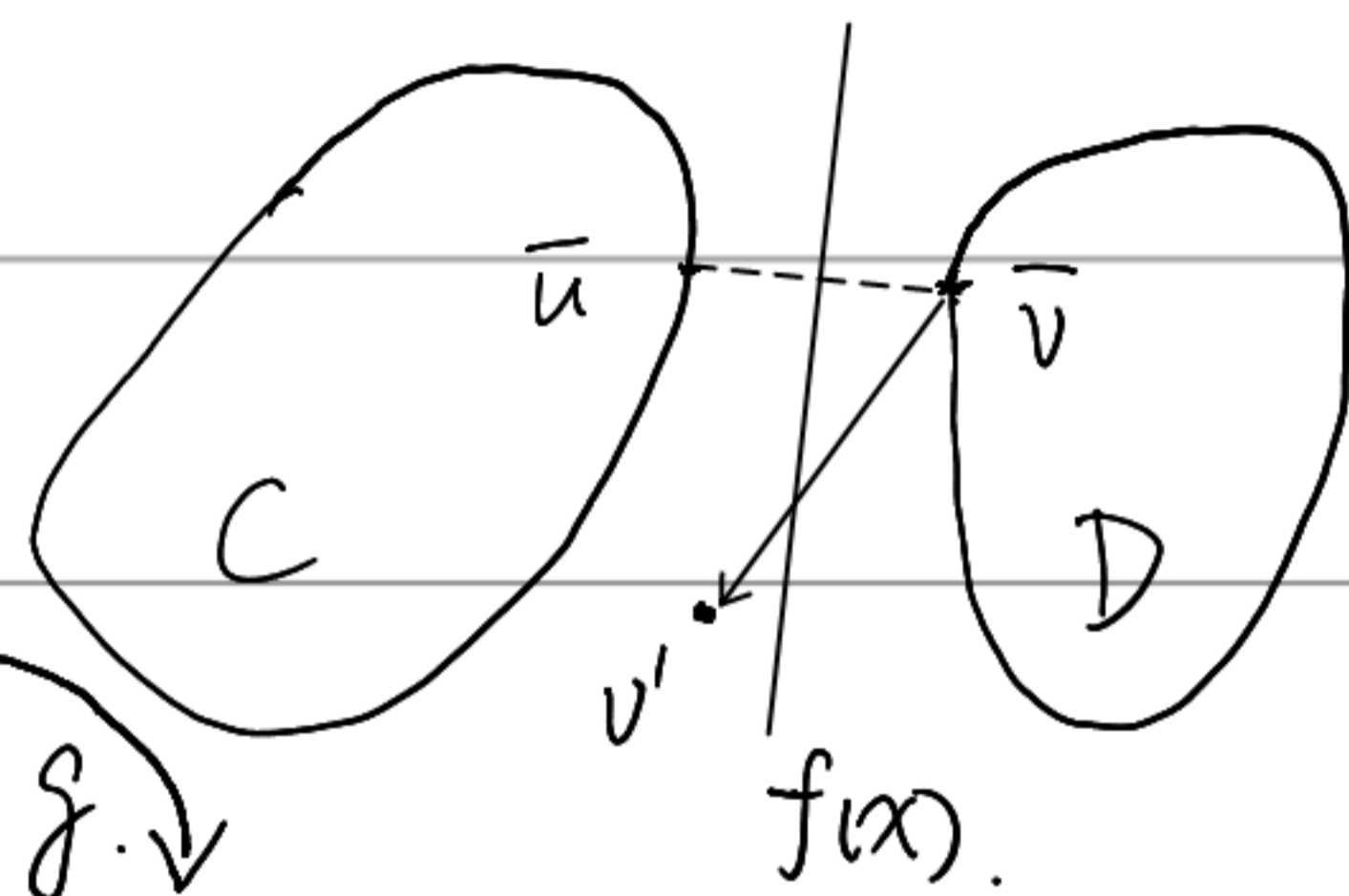
remark: why  $b$  is chosen?  $f\left(\frac{\bar{u} + \bar{v}}{2}\right) = (\bar{v} - \bar{u})^T \left(\frac{\bar{u} + \bar{v}}{2}\right) + \frac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2} = 0$ .

suppose for the sake of contradiction that  $\exists v' \in D$  s.t.  $f(v') \leq 0$ .

$$\Rightarrow (\bar{v} - \bar{u})^T v' - \frac{\|\bar{v}\|^2 - \|\bar{u}\|^2}{2} \leq 0.$$

Define  $g(x) = \|x - \bar{u}\|^2$ .

claim:  $v' - \bar{v}$  is a descent direction of  $g$  at  $\bar{v}$ .



proof:  $\nabla g(\bar{v})^T (v' - \bar{v}) = 2(\bar{v} - \bar{u})^T (v' - \bar{v})$

$$\begin{aligned} \nabla g(x) &= \nabla (x - \bar{u})^T (x - \bar{u}) \\ &= \nabla (x^T x - 2\bar{u}^T x + \bar{u}^T \bar{u}) \\ &= 2(x - \bar{u}) \\ &= 2(-\|\bar{v}\|^2 + \underbrace{\bar{v}^T v'} + \underbrace{\bar{u}^T \bar{v} - \bar{u}^T v'}) \\ &= 2(-\|\bar{v}\|^2 + w^T v' + \bar{u}^T \bar{v}) \\ &\leq 2(-\|\bar{v}\|^2 - \frac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2} + \bar{u}^T \bar{v}) \leq 0 \\ &= -(\|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\bar{u}^T \bar{v}) = -\|\bar{u} - \bar{v}\|^2 \end{aligned}$$

Thus  $\exists \theta > 0$  s.t.  $g(\bar{v} + \theta(v' - \bar{v})) < g(\bar{v})$  for  $\forall \theta' < \theta$ .

by convexity. If  $\theta < 1, \bar{v} + \theta(v' - \bar{v}) \in D \Rightarrow$  contradiction.  $\square$

corollary 1. If  $C \subseteq \mathbb{R}^n$  closed and convex.  $d \in \mathbb{R}^n \setminus C$ .

Then  $C$  and  $d$  can be strictly separated by a hyperplane.

corollary 2. supporting hyperplane theorem.

suppose  $C \subseteq \mathbb{R}^n$  ball  $B(x, \varepsilon) \triangleq \{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$ .

interior :  $\text{int } C \triangleq \{x \in C : \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \subseteq C\}$ .

closure :  $\text{cl } C \triangleq \{x \in \mathbb{R}^n : \exists x_0, \dots, x_n \in C \text{ s.t. } \lim x_n \rightarrow x\}$ .

(by definition) if  $C$  open  $\text{int } C = C$ . closed.  $\text{cl } C = C$ .

boundary :  $\text{bd } C$ , or  $\partial C \triangleq \text{cl } C \setminus \text{int } C$ .

SHT :  $\forall C \neq \emptyset$  is convex.  $x_0 \in \partial C \exists w^T \neq 0$  s.t.

$$\forall x \in C, w^T x \leq w^T x_0.$$

where  $\{x : w^T x = w^T x_0\}$  is called a supporting hyperplane of  $C$  at  $x_0$  ↓

Proof: If  $\text{int } C = \emptyset$ .  $C$  lies in an affine set of  $\text{dim} < n$ .

otherwise  $\exists n+1$  affinely independent points in  $C$

$C$  contains an  $n$ -simplex and thus interior points.

if  $\text{int } C \neq \emptyset$ . let  $C_{-\varepsilon} \triangleq \{x : B(x, \varepsilon) \subseteq C\}$ .

$\forall \varepsilon > 0, \exists w_\varepsilon \neq 0$  s.t.  $w_\varepsilon^T x < w_\varepsilon^T x_0$  for  $\forall x \in C_{-\varepsilon}$ .



w.l.o.g let  $\|w_\varepsilon\| = 1$ . let  $\varepsilon_n = \frac{1}{n} \rightarrow 0$ .

$\exists$  a subsequence of  $\{w_{\varepsilon_n}\}$  converge to  $w$ .

$\forall x \in \text{int } C$ .  $\exists N > 0$ . s.t.  $\forall n > N$ .  $w_{\varepsilon_n}^T x < w_{\varepsilon_n}^T x_0$

$\Rightarrow w^T x \leq w^T x_0 \quad \forall y \in \partial C$ .  $\exists \{x_n \in C\} \rightarrow y$ .

$\Rightarrow w^T y \leq w^T x_0 \quad \Rightarrow \forall v \in C \quad w^T v \leq w^T x_0$ .  $\square$ .

Finally. prove separating hyperplane theorem.

$\forall C, D \subseteq \mathbb{R}^n$ . are two convex sets.  $\exists w^T \neq 0$  and  $b$ . s.t.

$\forall u \in C$ .  $\forall v \in D$ .  $w^T u \leq b \leq w^T v$ .

Proof: consider  $C-D \triangleq \{u-v : u \in C, v \in D\}$ .

$C-D$  is also convex. separating  $C-D$  and  $\{0\}$

Goal: find  $w \neq 0$ . s.t.  $w^T x \leq 0$  for  $\forall x \in C-D$ .

Case 1:  $0 \notin \partial(C-D)$  strictly separating hyperplane theorem:

$\exists w \neq 0$  separating (strictly)  $\text{cl}(C-D)$  and  $0$

$\Rightarrow \forall u \in C$ .  $\forall v \in D$ .  $w^T(u-v) \leq 0$

Case 2:  $0 \in \partial(C-D)$ . supporting hyperplane theorem.

$\exists w \neq 0 \quad \forall u \in C$ .  $\forall v \in D$ .  $w^T(u-v) \leq 0$ .  $\square$

