

## Lecture 6. Convex functions.

more about separating and supporting hyperplanes.

① why did we emphasize "finite" halfspaces when define polytope.

any closed convex set can be expressed as intersection of a family of halfspaces: all supporting hyperplanes.

proof:  $C \subseteq$  intersection trivial.

$C \supseteq$  intersection  $\forall v \notin C$ . suppose  $u \in C$ . s.t.

$\|u - v\| = \text{dist}(C, v)$ . separating  $C$  and  $\{v\}$  at  $u$ .

② converse theorem of separating hyperplanes.

convex disjoint  $\} \Rightarrow$  separating.

Thm. any convex sets  $C$  and  $D$ . at least one of which is open.

$C, D$  disjoint iff there exists a separating hyperplane.

Proof:  $\Leftarrow$  w.l.o.g. assume  $C$  is open and  $f(x) = w^T x + b$ .

s.t.  $f(x) \leq 0$  for all  $x \in C$ .  $f(x) \geq 0$  for all  $x \in D$ .

if  $C \cap D \neq \emptyset$ .  $\forall x \in C \cap D$ .  $f(x) = 0$ .

contradiction.

$C$  is open  $\Rightarrow \exists \varepsilon > 0$ .  $B(x, \varepsilon) \subseteq C$ .  $\Rightarrow \exists y \in C$ .  $\overbrace{f(y)} > 0$ .

③ converse theorem of supporting hyperplane.

Thm.  $\forall$  closed set  $C \subseteq \mathbb{R}^n$  has nonempty interior.

$C$  has a supporting hyperplane at any  $x \in \partial C \Rightarrow$  convex.

New topic: convex functions.

Def. (Jensen's inequality). a function  $f: D \rightarrow \mathbb{R}$  is convex.

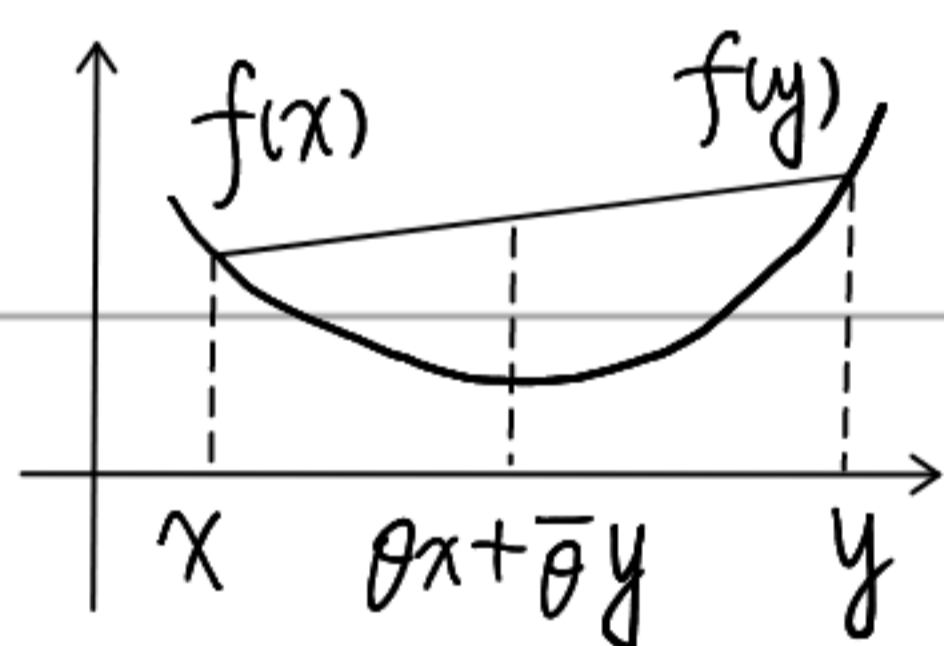
if  $\forall x, y \in D \triangleq \text{dom } f$ , and  $\forall \theta \in (0, 1)$ .

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y).$$

strictly convex: if  $f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$

We say  $f$  is concave if  $-f$  is convex.

strictly concave if  $-f$  is strictly convex.



a technical point: domain of  $f$

$D$  must be a convex set.

Extended-value extension: for convenience, extend  $\mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ .

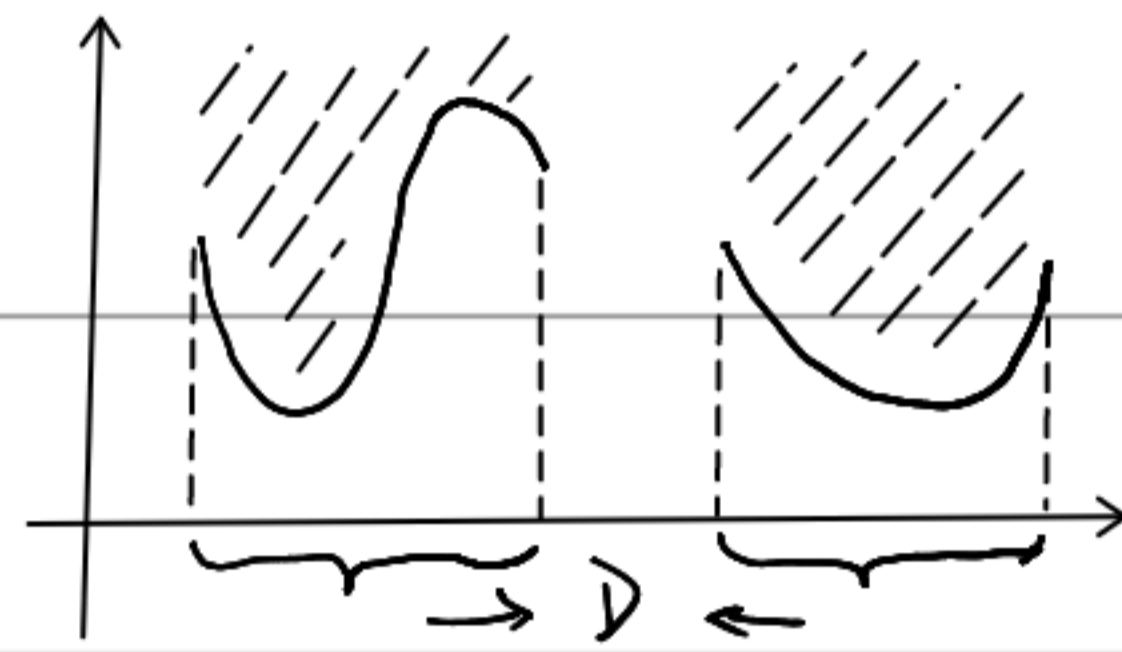
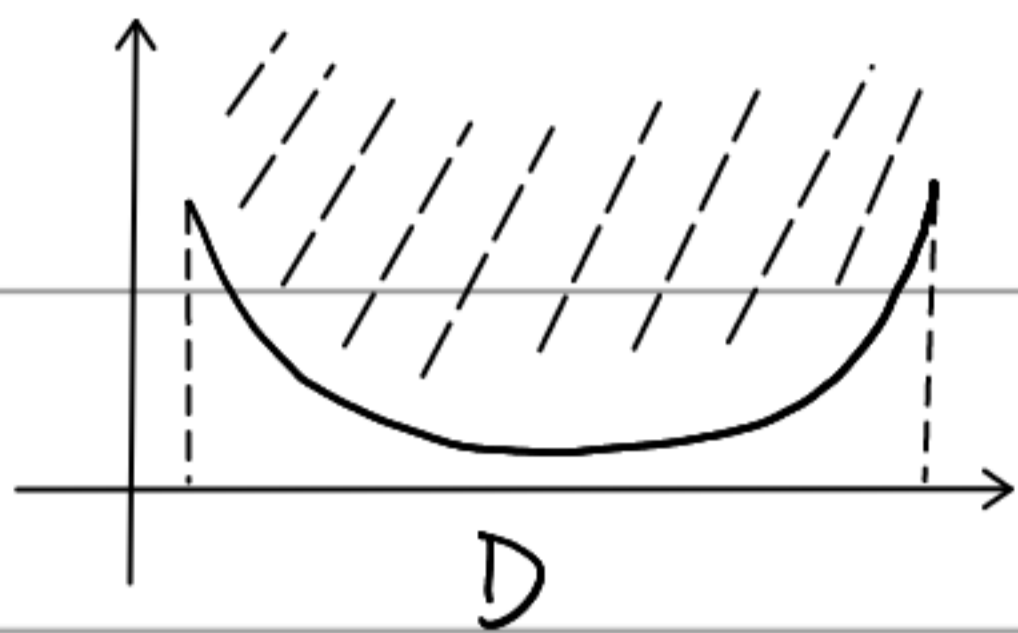
$$\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\} = \begin{cases} f(x) & x \in D \\ \infty & x \notin D. \end{cases} \quad D = \text{dom } f.$$

where by convention.  $x + \infty = \infty$ .  $x \cdot \infty = \infty$ .  $0 \cdot \infty = 0$ .

Example:  $f(x) = \begin{cases} 1/x & x > 0 \\ \infty & x \leq 0 \end{cases}$  is a convex function.

Epigraph:  $f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . graph  $\{(\vec{x}, f(\vec{x})) \in \mathbb{R}^{n+1} : \vec{x} \in D\}$ .

epigraph of  $f$ :  $\text{epi } f = \{(\vec{x}, y) \in \mathbb{R}^{n+1} : \vec{x} \in D, y \geq f(\vec{x})\}$



Thm:  $f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function iff  $\text{epi } f$  convex.

Proof:  $\Rightarrow$ :  $f$  is convex. goal:  $\text{epi } f$  is convex.

Given  $(x_1, y_1), (x_2, y_2) \in \text{epi } f$ .  $\theta \in [0, 1]$ .

let  $(x, y) = (\theta x_1 + \bar{\theta} x_2, \theta y_1 + \bar{\theta} y_2)$ .

$f$  convex  $\Rightarrow x \in D$ .  $f(x) \leq \theta f(x_1) + \bar{\theta} f(x_2)$ .

$(x_i, y_i) \in \text{epi } f \Rightarrow f(x_i) \leq y_i \Rightarrow f(x) \leq \theta y_1 + \bar{\theta} y_2 = y$ .

$\Leftarrow$ :  $\text{epi } f$  is convex. goal:  $f$  is convex. (first,  $D$  is convex).

Given.  $x_1, x_2 \in D$ .  $\theta \in [0, 1]$ . let  $x = \theta x_1 + \bar{\theta} x_2$

$(x_1, f(x_1)), (x_2, f(x_2)) \in \text{epi } f \Rightarrow (x, \theta f(x_1) + \bar{\theta} f(x_2)) \in \text{epi } f$

$\Rightarrow x \in D$ , and  $\theta f(x_1) + \bar{\theta} f(x_2) \geq f(x)$ .  $\square$ .

Examples of convex functions:  $e^{ax}$ ,  $-\log x$ ,  $x^a$  if  $a \geq 1$  or  $a \leq 0$ .  
 affine functions are convex and concave.  $\rightarrow$  strictly if  $a \neq 0, 1$ .  
 not strictly.

