

## Lecture 8. Inequalities. Optimization problems.

Example of composition: log-sum-exp. norm  $(a, b)$  of matrices.

$$\|A\|_{a,b} = \sup_{w \neq 0} \frac{\|Aw\|_a}{\|w\|_b} \quad f_w(A) = \|Aw\|_a \text{ convex. } \|\cdot\|_{a,b} = \sup_{\|w\|_b=1} f_w(A).$$

Jensen's inequality:  $f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y)$  convex.

generalization:  $f(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k) \leq \theta_1 f(x_1) + \theta_2 f(x_2) + \dots + \theta_k f(x_k)$

where  $\theta_1, \theta_2, \dots, \theta_k \geq 0$  and  $\theta_1 + \theta_2 + \dots + \theta_k = 1$ .

integrals: if  $p(x) \geq 0$  defined on  $\Omega \subseteq \text{dom } f$ .  $\int_{\Omega} p(x) dx = 1$  then.

$f(\int_{\Omega} p(x) x dx) \leq \int_{\Omega} f(x) p(x) dx$ . provided the integrals exist.

probability: if  $x$  is a random variable. s.t.  $\Pr[x \in \text{dom } f] = 1$ . then

$f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$ . provided the expectations exist.

Recall Cauchy-Schwarz.  $\langle x, y \rangle \leq \|x\|_2 \|y\|_2$ .

$\langle x, y \rangle \leq \|x\|_3 \|y\|_3$   $\|x\|_3 \leq \|x\|_2$  ?

proposition. given  $1 \leq p_1 < p_2 \leq \infty$ .  $x_i \geq 0$ .  $\|x\|_{p_1} \geq \|x\|_{p_2}$ .

Proof. if  $p_2 = \infty$ .  $\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i| \leq (\sum |x_i|^{p_1})^{1/p_1}$ .

let  $g = (\sum |x_i|^{p_1})^{1/p_1}$   $\tilde{x}_i = x_i / g$ .  $\|\tilde{x}\|_{p_1} = 1$ .

$\|\tilde{x}\|_{p_2} = (\sum_{i \in [0,1]} (|\tilde{x}_i|^{p_1})^{p_2/p_1})^{1/p_2} \leq (\sum |\tilde{x}_i|^{p_1})^{1/p_2} = 1$ .  $\square$

Hölder's inequality: let  $p, q \in (1, \infty)$  be conjugate exponents.

i.e.  $1/p + 1/q = 1$ . then  $\langle x, y \rangle \leq \|x\|_p \|y\|_q \quad \forall x, y \in \mathbb{R}^n$ .

$$\text{or } \sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \left( \sum_{i=1}^n |y_i|^q \right)^{1/q}.$$

Remark: let  $p = q = 2$ . it is Cauchy-Schwarz.

Proof. w.l.o.g. assume  $x_i \geq 0, y_i \geq 0$  and  $\langle x, y \rangle > 0$ .

let  $\tilde{x} = x / \|x\|_p, \tilde{y} = y / \|y\|_q$ . goal:  $\sum |\tilde{x}_i \tilde{y}_i| \leq 1$ .

We first show that  $u^{1/p} v^{1/q} \leq u/p + v/q \quad \forall u, v \geq 0$ .

if  $uv = 0$ . trivial. o.w.  $\log(u/p + v/q) \geq \frac{1}{p} \log u + \frac{1}{q} \log v$ .

$$\Rightarrow |\tilde{x}_i| \cdot |\tilde{y}_i| \leq \frac{1}{p} |\tilde{x}_i|^p + \frac{1}{q} |\tilde{y}_i|^q \Rightarrow \sum |\tilde{x}_i \tilde{y}_i| \leq \frac{1}{p} + \frac{1}{q} = 1. \quad \square$$

Minkowski's inequality: verify triangle inequality for  $L_p$ -norms.

$$\text{for } 1 \leq p \leq \infty \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Proof. only consider  $\|x+y\|_p > 0$ . (the right hand side is nonnegative).

$$\|x+y\|_p^p = \sum |x_i + y_i|^p \leq \sum |x_i| \cdot |x_i + y_i|^{p-1} + \sum |y_i| \cdot |x_i + y_i|^{p-1}$$

By Hölder's inequality  $\langle u, v \rangle \leq \|u\|_p \|v\|_{p/(p-1)}$

$$\Rightarrow \sum |x_i| \cdot |x_i + y_i|^{p-1} \leq \|x\|_p \left( \sum |x_i + y_i|^p \right)^{(p-1)/p} = \|x\|_p \|x+y\|_p^{p-1}$$

$$\Rightarrow \|x+y\|_p^p \leq (\|x\|_p + \|y\|_p) \|x+y\|_p^{p-1} \Rightarrow \text{Minkowski's ineq.} \quad \square$$

# Convex Optimization Problems

standard form: 
$$\begin{aligned} \min & f(x) && \text{objective function} \\ \text{s.t.} & g_i(x) \leq 0, && i=1, 2, \dots, m. \\ & h_i(x) = 0, && i=1, 2, \dots, k. \end{aligned}$$

domain of problem (P) is

constraint function

$$D \triangleq \text{dom } f \cap \left( \bigcap_{i=1}^m \text{dom } g_i \right) \cap \left( \bigcap_{i=1}^k \text{dom } h_i \right).$$

the feasible set is (P is feasible if  $\Omega \neq \emptyset$ ).

$$\Omega \triangleq \{x \in D : g_i(x) \leq 0, 1 \leq i \leq m; h_i(x) = 0, 1 \leq i \leq k\}$$

the optimal value of P is  $f^* \triangleq \inf_{x \in \Omega} f(x)$ .

Remark. allow  $f^*$  to take the extended values  $\pm \infty$ .

-  $f^* = \infty$  if P is infeasible. i.e.  $\Omega = \emptyset$ .  $\sup \phi = -\infty$   
 $\inf \phi = \infty$ .

-  $f^* = -\infty$  if  $f(x)$  is unbounded below.

-  $x^*$  is an optimal point solving (P). if  $x^* \in \Omega$  s.t.  $f(x^*) = f^*$

-  $x^*$  is a locally optimal if  $f(x^*) \leq f(x), \forall \|x - x^*\| < \delta$   
for some  $\delta > 0$

optimization  $\rightarrow$  convex optimization:  $f, g_i$  convex.  $h_i$  affine.

domain:  $D = \text{dom } f \cap \left( \bigcap \text{dom } g_i \right)$  ——— convex sets.

feasible set:  $\Omega = \{x \in D : g_i(x) \leq 0, h_j(x) = 0\}$   
 $\alpha$ -sublevel sets. ——— minimizing convex functions over convex sets.

