Lecture 9. Linear programming.

Linear program: $\min_x c^T x$ s.t. $Bx \le d$. $Ax = b$.

Standard form: $\min_x c^T x$ s.t. $Ax = b$. $x \ge 0$.

- adding slack variables. s. $\min_{x,s} c^T x$. s.t. $Bx + s = d$. $s \ge 0$.
- splitting variables into positive and negative parts $x = x^+ - x^-$.

$$\min_{x^+, x^-, s} c^T x^+ - c^T x^- \quad \text{s.t. } Bx^+ - Bx^- + s = d.\ Ax^+ - Ax^- = b.\ x^+, x^-, s \ge 0$$

Example. $\min -2x_1 - 3x_2$ s.t. $x_1 \le 100$. $x_2 \le 200$. $x_1 + x_2 \le 160$.

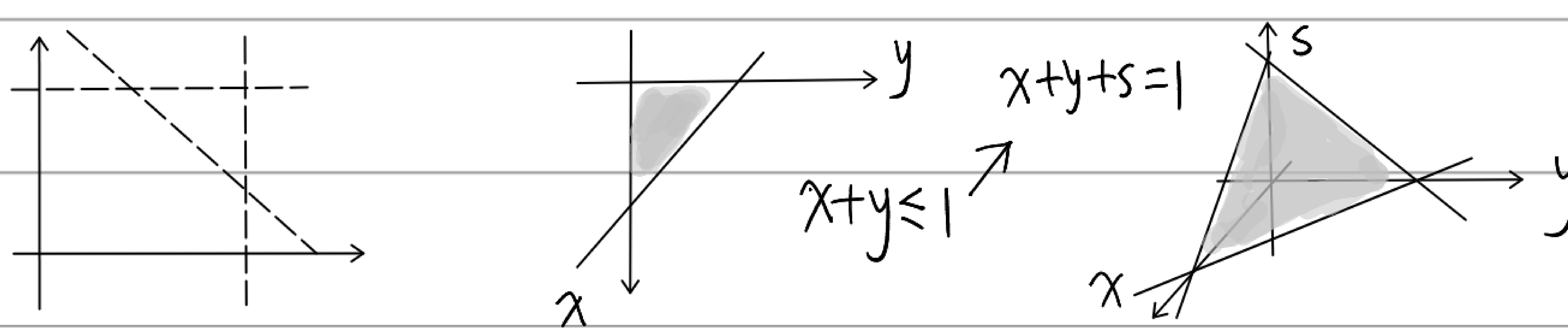
step 1. $\min -2x_1 - 3x_2$. s.t. $x_1 + s_1 = 100$. $x_2 + s_2 = 200$ $x_1 + x_2 + s_3 = 160$.

$s_1, s_2, s_3 \ge 0$.

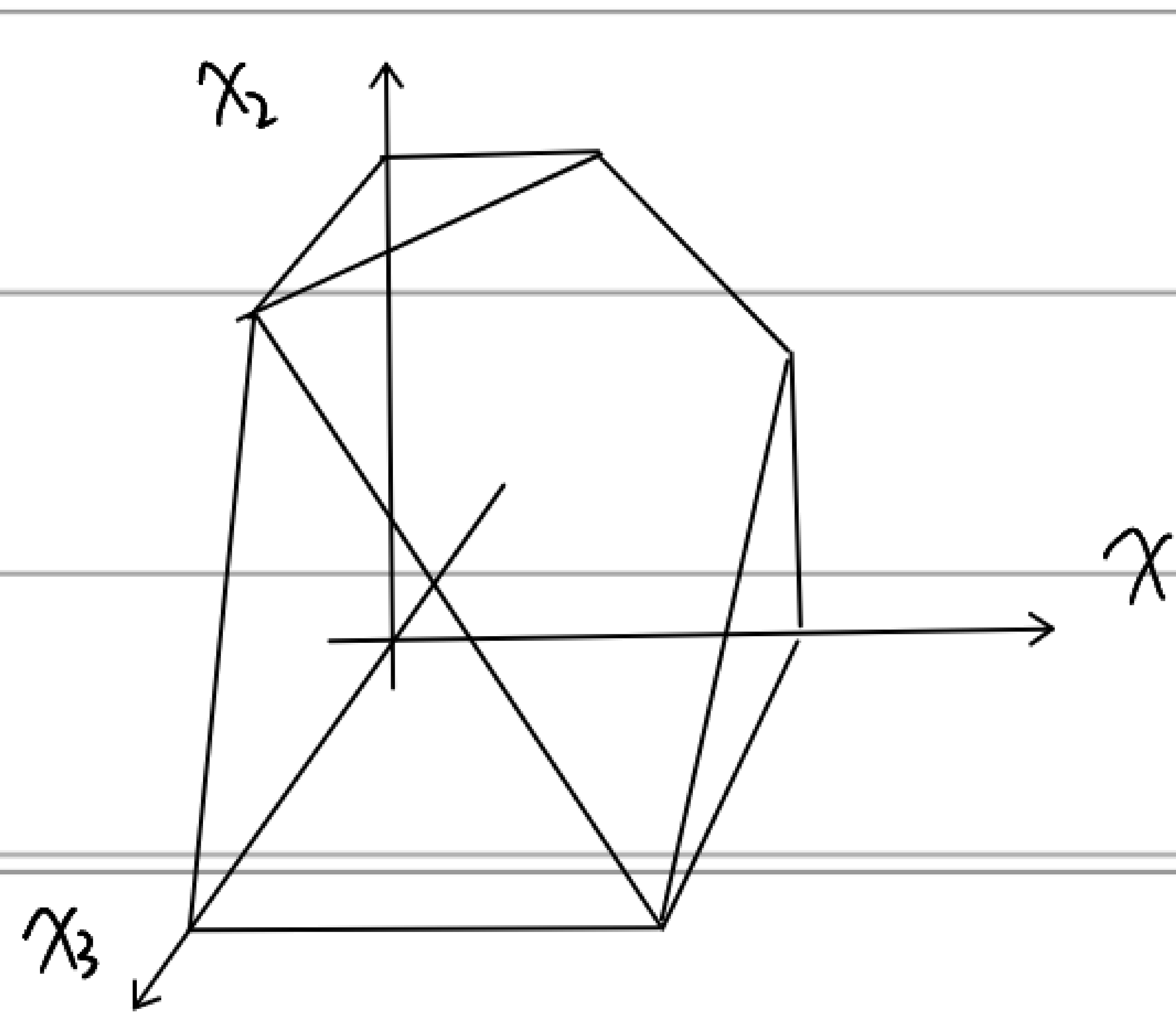
step 2. $\min -2(x_1^+ - x_1^-) - 3(x_2^+ - x_2^-)$

s.t. $(x_1^+ - x_1^-) + s_1 = 100$, $(x_2^+ - x_2^-) + s_2 = 200$.

$(x_1^+ - x_1^-) + (x_2^+ - x_2^-) + s_3 = 160$. $x_1^+, x_1^-, x_2^+, x_2^-, s_1, s_2, s_3 \ge 0$.



Example. $\min -x_1 + 6x_2 - 13x_3$



$x_1 + x_2 + x_3 \le 400$

$x_2 + 3x_3 \le 600$.

$x_1 \le 200$, $x_2 \le 300$

x_1 x_2 $x_3 \ge 0$.

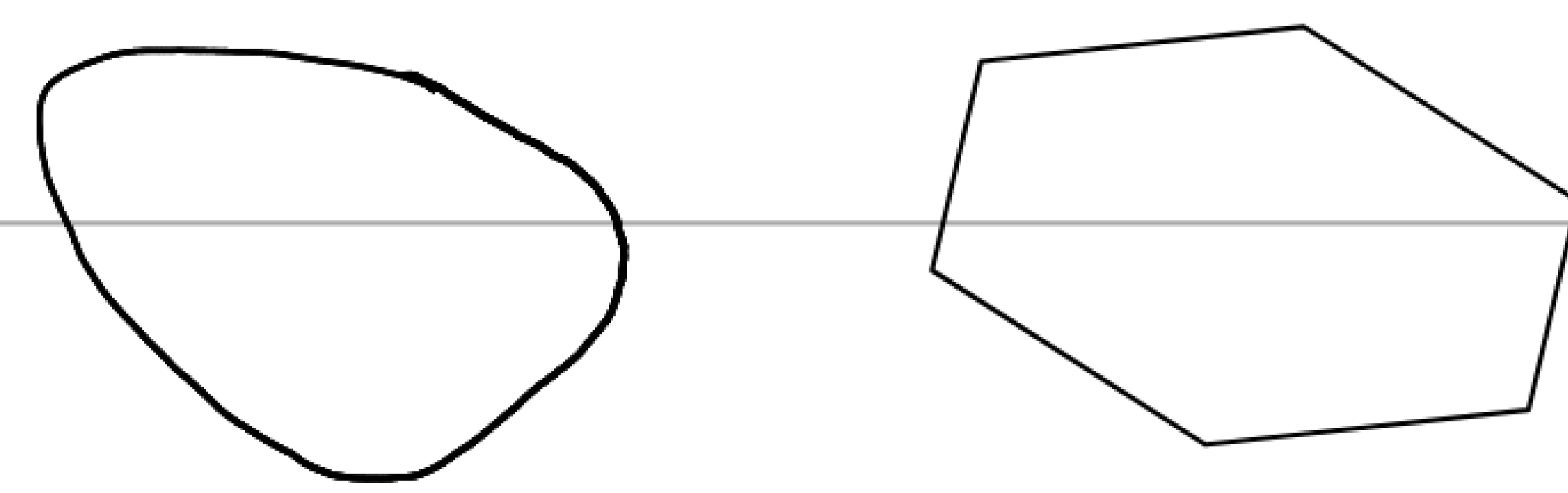
All possibilities for an LP: infeasible / unbounded / $\exists$ optimal solutions

optimal : infinite number of optimal solutions / unique optimal solution

Conjecture : $\exists$ an optimal solution at a vertex if there exists optimal.

Extreme points : $x \in C$ (convex) is an extreme point if x is not a convex combination of two other points. i.e. $\nexists\, \theta, y, z$ s.t. $x = \theta y + \bar{\theta} z$, $y \neq z$.

what is / are extreme points?



Vertex : $x \in \mathbb{R}^n$ is a vertex of polyhedron P if $x \in P$ and $\exists\, n$ linear independent constraints that are tight at x.

Basic solution : $Ax = b$. $A \in \mathbb{R}^{m \times n}$. $\exists\, m$ linearly independent columns.

$A = (\underset{\in \mathbb{R}^{m\times m}}{B}, D)$. $x = (x_B^T, 0^T)^T$ basic solution. $x \geq 0$. basic feasible sol.

Another view by slackness. n variables. m constraints n slack variable.

$Ax \leq b \rightarrow A'x' = b$ $\quad \mathbb{R}^{m\times n} \rightarrow \mathbb{R}^{m \times (n+m)}$ $\quad x_B \in \mathbb{R}^m$. $0 \in \mathbb{R}^n$

vertex : n tight constraints $\Rightarrow$ n slack variable $= 0$

Equivalence of extreme point and vertex :

"$\Leftarrow$". $x \in P$ be a vertex. $\Rightarrow$ n linear independent constraints tight at x.

$\Rightarrow \exists \tilde{A} \in \mathbb{R}^{n\times n}, \tilde{b} \in \mathbb{R}^n$. s.t. $\tilde{A}x = \tilde{b}$. Suppose $x = \theta y + \bar{\theta} z$, $y, z \in P$

$\Rightarrow \theta \tilde{A} y + \bar{\theta} \tilde{A} z = \tilde{b}$. $\tilde{A}y$, $\tilde{A}z \le b \Rightarrow \tilde{A}y = \tilde{A}z = b$. but $\tilde{A}$ invertible.

"$\Rightarrow$". suppose $x \in P$ not a vertex. Let $I = \{i : a_i^T x = b_i\}$.

so there does not exist n linearly independent a_i s.t. $i \in I$.

$a_i \in \mathbb{R}^n \Rightarrow \exists \overset{\neq 0}{d} \in \mathbb{R}^n$, $d^T a_i = 0$. $\forall i \in I$. let $y = x + \varepsilon d$, $z = x - \varepsilon d$.

$\forall i \in I$. $a_i^T y = a_i^T z = b$. since $a_i^T d = 0$. $\forall i \notin I$. $f(w) = b_i - a_i^T w$

continuous. and $f(x) > 0$. choose ε sufficiently small. $f(y)$. $f(z) \geq 0$ $\square$

Corollary. given a finite set of linear inequalities, only finite extreme points.

Existence of extreme points: iff P does not contain a line.

Corollary: every bounded polyhedron (every polytope) has an extreme.

Fundamental theorem of linear programming.

consider the linear program. $\min c^T x$. s.t. $Ax \le b$. $x \geq 0$

suppose P has ≥ 1 optimal solution. then $\exists$ an optimal sol at a vertex.

Algorithmic application: find optimal solution by enumerating $\binom{m}{n}$ vertices.

However. $\{0, 1\}^n$ - cube:

2n inequalities.
but 2^n vertices.