Lecture 4. Affine and convex sets

Affine combination: $\forall \theta_i \in \mathbb{R}$. $\theta_1 + \cdots + \theta_n = 1$. $\underbrace{\theta_1 x_1 + \cdots + \theta_n x_n}$

Affine set: given $x_1, \dots, x_n \in S$. $\forall$ affine combination $\in S$.

Example. solution to a linear equation $S = \{x : Ax = b\}$.

$(Ax_1 = Ax_2 = b \Rightarrow \forall \theta_1 + \theta_2 = 1.\ A(\theta_1 x_1 + \theta_2 x_2) = \theta_1 A x_1 + \theta_2 A x_2 = b)$.

Conversely. any affine set is the solution to a linear equation set.

Why? $S' = S - x_0$ is a linear space. $\forall x_1, x_2 \in S'$ $a_1 x_1 + a_2 x_2 \in S'$.

$$\begin{matrix} x_1 + x_0 \\ x_2 + x_0 \end{matrix} \in S \Rightarrow a_1 x_1 + a_2 x_2 + x_0 = a_1(x_1 + x_0) + a_2(x_2 + x_0) + (1 - a_1 - a_2) x_0 \in S$$

In particular. if $A \in \mathbb{R}^{1 \times n}$. $S = \{x : w^T x = b\}$ is a hyperplane.

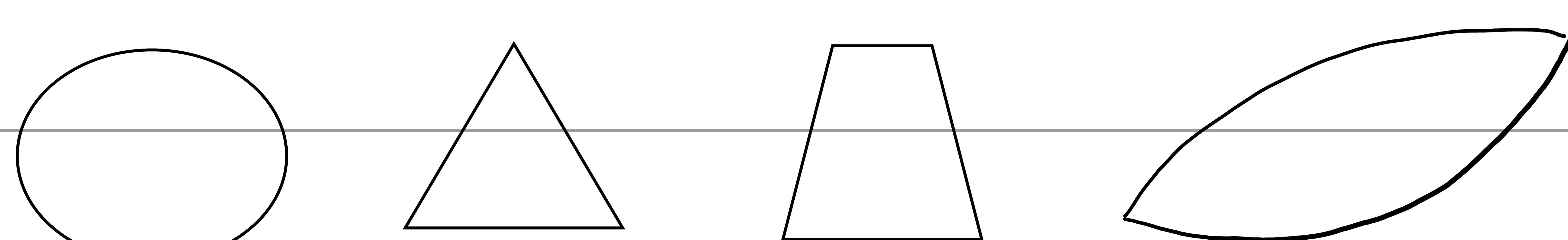
If $A \neq 0$. affine set $S \neq \mathbb{R}^n$ is the intersection of finite hyperplanes.

line: $z = x + \theta(y - x) = (1-\theta)x + \theta y$. $\theta \in \mathbb{R} \longrightarrow$ affine sets

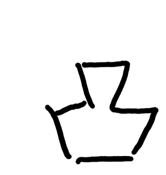
segment: $z = x + \theta(y - x) = (1-\theta)x + \theta y$. $\theta \in [0,1] \longrightarrow$? convex sets.

Convex combination: $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$. $\forall \theta_i \geq 0$. $\sum \theta_i = 1$.

Convex sets: if $x_1, \dots, x_n \in S$, any convex combination $\in S$. ($n=2$ suffices)

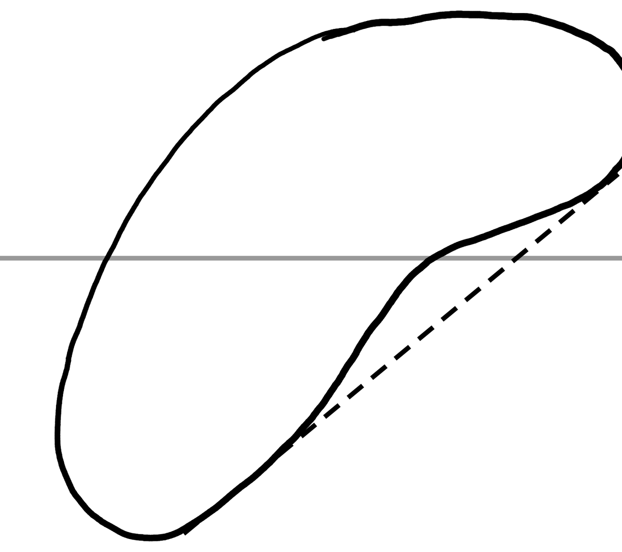


in particular

 is not convex.

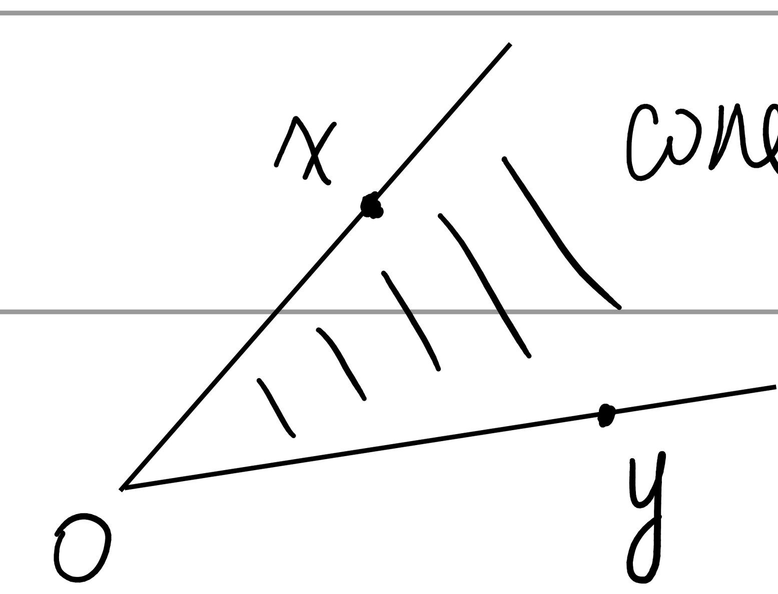
Convex hull: set of all convex combinations of points in S.

$\text{conv}(S) \triangleq \{ \sum \theta_i x_i : \theta_i \geq 0, \ \sum \theta_i = 1, \ x_i \in S \}$.



Conic combination: $z = \theta_1 x + \theta_2 y$. $\theta_1, \theta_2 \geq 0$.

convex cone: set of all conic combinations of points in S.



Examples of convex sets:

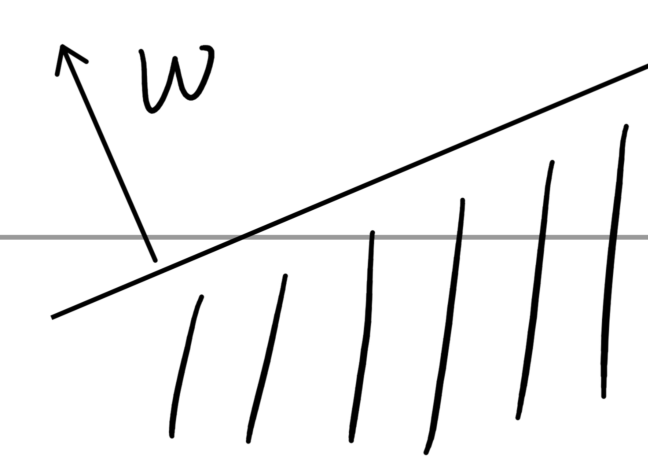
$\mathbb{R}^n$. affine sets hyperplanes

hyperplane $\{ x : w^T x + b = 0 \}$. halfspace $\{ x : w^T x + b \ \substack{\leq \\ \geq} \ 0 \}$.

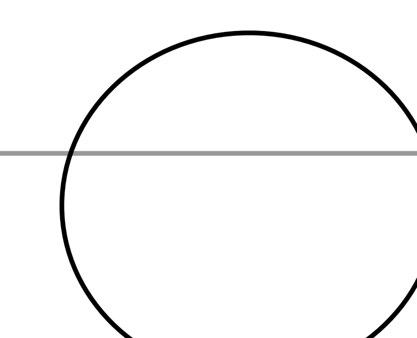
Halfspaces are convex. but not affine. open halfspaces also convex

$\forall x, y \in \{ x : w^T x + b \leq 0 \}$. let $\bar{\theta} = 1 - \theta$

$w^T(\theta x + \bar{\theta} y) + b = \theta(w^T x + b) + \bar{\theta}(w^T x + b) \leq 0$.



Euclidean balls and ellipsoids are convex.



Ball: $\{ x : \|x - x_0\| \leq r \}$. $\forall x, y \in \mathcal{B}(x_0, r)$.

$\|\theta x + \bar{\theta} y - x_0\| = \|\theta(x - x_0) + \bar{\theta}(y - x_0)\| \leq \theta \|x - x_0\| + \bar{\theta} \|y - x_0\|$.

Ellipsoid: $\{ (x_1, x_2)^T : \frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} \leq 1 \}$. convex. why? homework.

Proposition: The image of a convex set under an affine transform is also convex. i.e., $C \subseteq \mathbb{R}^n$ convex $\Rightarrow AC + b = \{ Ax + b : x \in C \}$ convex.

Proof. Let $f(x) = Ax + b$. $\forall y_1, y_2 \in f(C)$. $\exists x_i$. $y_i = f(x_i)$.

$\theta y_1 + \bar{\theta} y_2 = \theta(Ax_1 + b) + \bar{\theta}(Ax_2 + b) = A(\underbrace{\theta x_1 + \bar{\theta} x_2}_{\in C \text{ by convexity}}) + b \in f(C)$. $\square$

Question: How about inverse image? (homework)

Proposition: some other convexity-preserving operations.

- intersection: if C and D are convex. then so is $C \cap D$.
- Cartesian product: $C \times D \triangleq \{(x_1, x_2): x_1 \in C, x_2 \in D\}$ uncountably infinite many intersection ✓
- set sum (Minkowski addition): $C + D \triangleq \{x_1 + x_2 : x_1 \in C, x_2 \in D\}$

A nongeometric example of convex sets: positive semidefinite matrices.

$S_+^n \triangleq \{A \in \mathbb{R}^{n \times n} : A \succeq 0\}$ $S_{++}^n \triangleq \{A \in \mathbb{R}^{n \times n} : A \succ 0\}$ convex.

Proof: $\forall A_1\ A_2 \in S_+^n$. A_1. A_2 symmetric $\Rightarrow \theta A_1 + \bar{\theta} A_2$ symmetric

$\forall v$. $v^T A_1 v \geq 0$. $v^T A_2 v \geq 0 \Rightarrow v^T(\theta A_1 + \bar{\theta} A_2) v \geq 0$ $\square$

Polyhedron (polyhedra) 多面体. polytope 多胞体.

A polyhedron is the intersection of some halfspaces.

Halfspace: $\{x : w^T x + b \leq 0\}$. Polyhedron: $\{x : \forall i.\ w_i^T x + b_i \leq 0\}$.

affine sets、halfspaces $\subseteq$ polyhedra (may unbounded) $\subseteq$ convex sets.

A polytope is an bounded polyhedron. Feasible sets of LP $\subseteq$ polyhedra

Simplex (simplices / simplexes) 单纯形 : simplest polytope.

n-dimensional simplex : convex hull of $(n+1)$ nondegenerate points.

Examples : 0-simplex : point 1-simplex : line segment.

2-simplex : triangle 3-simplex : tetrahedron.

nondegenerate ? $n+1$ affinely independent points in $\mathbb{R}^n$.

$u_0 = \theta_1 u_1 + \cdots + \theta_n u_n$. $\sum \theta_i = 1 \iff \theta_1(u_1 - u_0) + \cdots + \theta_n(u_n - u_0) = 0$.

$\Updownarrow$ $\Updownarrow$

$u_0, u_1, \ldots u_n$ affinely dependent $u_1 - u_0, \ldots u_n - u_0$ linearly dependent

$(n+1)$-dimensional simplex $S = \{\theta_0 u_0 + \cdots + \theta_n u_n : \sum \theta_i = 1, \theta_i \geq 0\}$.

standard simplex. $u_0 = 0$, $u_1 \ldots u_n$ = unit vector of n dimensions.

let $y \triangleq (\theta_1, \ldots, \theta_n)^T \in \mathbb{R}^n$. $B = (u_1 - u_0, \ldots, u_n - u_0) \in \mathbb{R}^{n \times n}$ full rank.

$S = \{\theta_0 u_0 + \cdots + \theta_n u_n : \sum \theta_i = 1, \theta_i \geq 0\} = \{u_0 + By : \sum y_i \leq 1, y_i \geq 0\}$

B has full rank $\Rightarrow \exists A = B^{-1} \in \mathbb{R}^n$. $A(u_0 + By) = Au_0 + y$. $\forall y$.

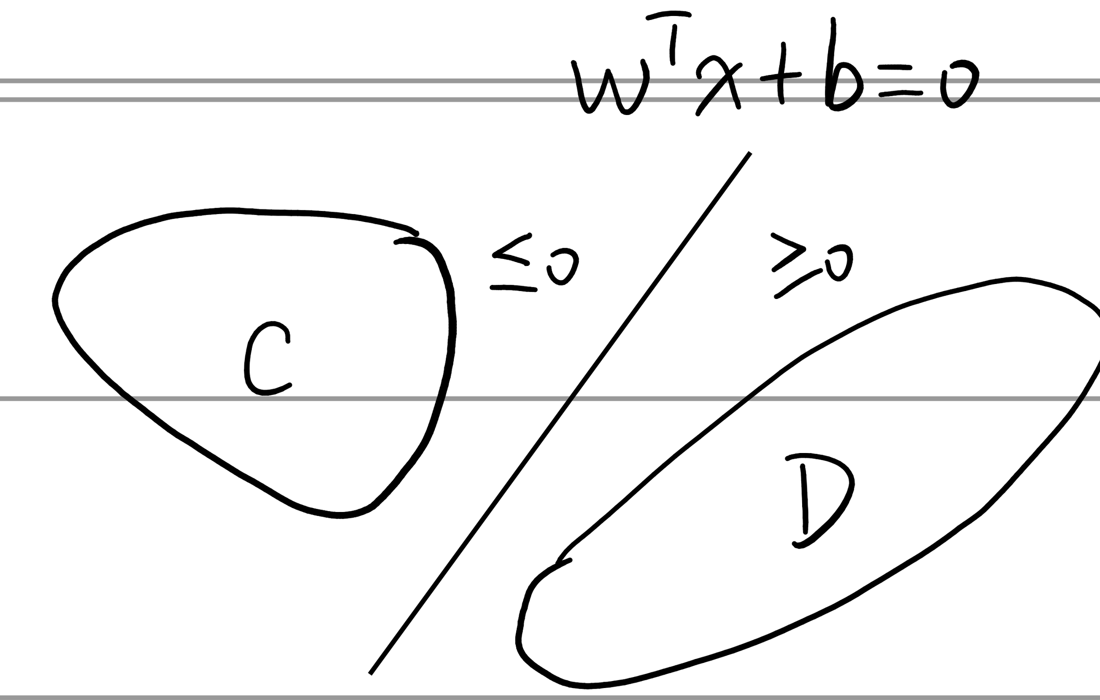
So $\forall x \in \mathbb{R}^n$, $x \in S$ iff $\exists y$, $x = u_0 + By$ iff $\exists y$. $Ax = Au_0 + y$.

$A = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$. $y_i \geq 0 \Rightarrow a_i^T x \geq a_i^T u_0$. $\sum y_i \leq 1 \Rightarrow \sum a_i^T (u_0 - x) \leq 1$.

Separating hyperplane theorem and supporting hyperplane theorem.

(One of the most important and fundamental property of convex sets.)

Separating hyperplane theorem.

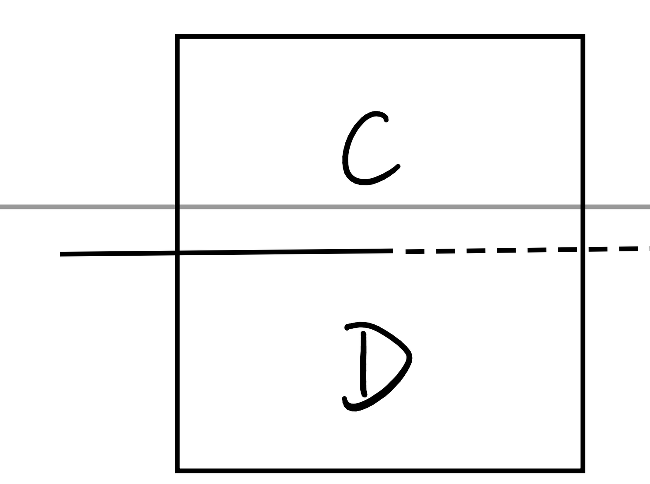


Let C, D be two disjoint convex sets.

Then $\exists$ hyperplane $w^Tx+b=0$ separating C and D. $\left(\begin{array}{l} w^Tx+b \le 0,\ x\in C \\ w^Tx+b \ge 0,\ x\in D \end{array}\right)$

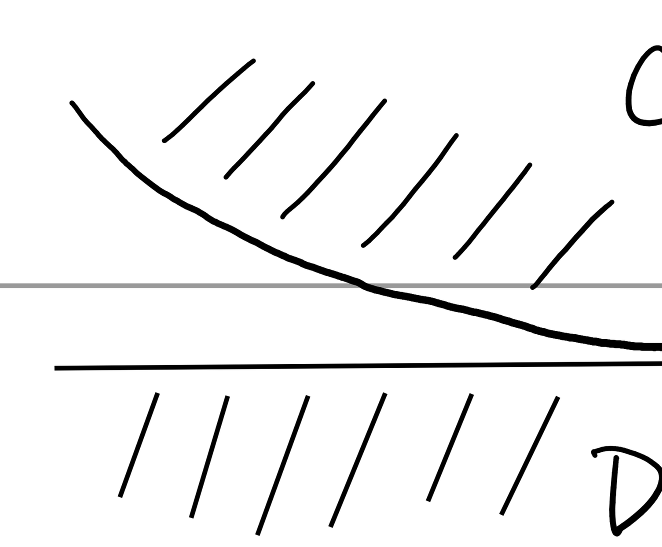
Strict separating theorem. (strict : $\begin{array}{l} w^Tx+b<0,\ x\in C \\ w^Tx+b>0,\ x\in D \end{array}$)

Let C, D be two disjoint closed convex sets.



s.t. at least one of them is bounded.

Then $\exists$ strict separting hyperplane $w^Tx+b=0$.

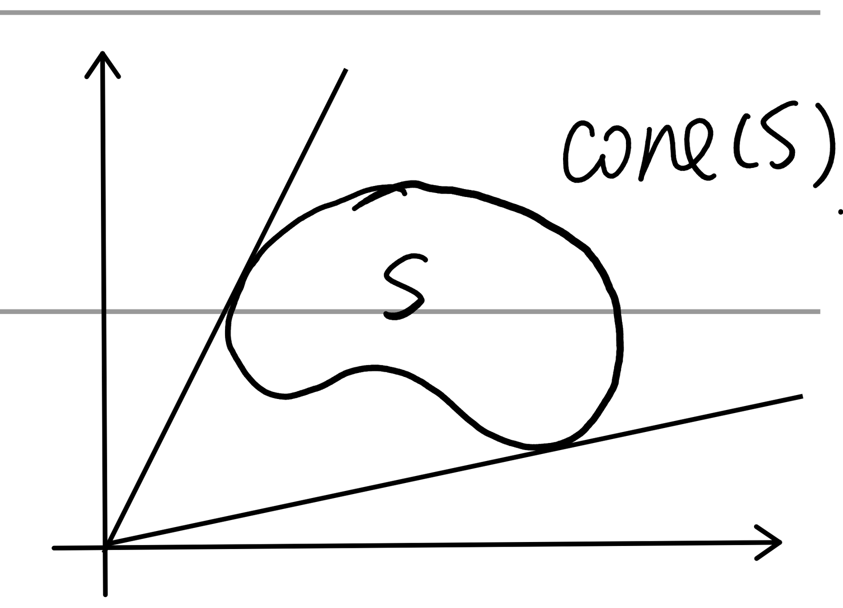


Application: Farkas' lemma. (LP duality).

Let $A\in\mathbb{R}^{n\times m}$, $b\in\mathbb{R}^n$. then exactly one of the following is ϕ:

① $\{x\in\mathbb{R}^m : Ax=b,\ x\ge\vec{0}\}$. ② $\{y\in\mathbb{R}^n : A^Ty\le 0,\ b^Ty>0\}$.

Proof. Recall the conic combination $\theta_1x_1+\cdots+\theta_kx_k$.



$\text{cone}(S) \triangleq \{\sum\theta_ix_i : \forall\theta_i\ge 0,\ x_i\in S\}$.

Let $A=(a_1,a_2,\dots,a_n)$. ① $=\phi \Rightarrow b\notin \text{cone}(A)$. closed. convex

By separating hyperplane (strict). $\exists y, z$. $\begin{array}{l} u^Ty+z<0 \quad \forall u\in\text{cone}(A) \\ b^Ty+z>0 \end{array}$

$\forall a_i$, $\forall\lambda_i\ge 0$. $\lambda_i a_i^Ty+z<0 \Rightarrow a_i^Ty+\frac{z}{\lambda_i}<0 \Rightarrow a_i^Ty\le 0$.

$0\in\text{cone}(A) \Rightarrow z<0 \Rightarrow b^Ty>0$. so y is a desired one. $\square$