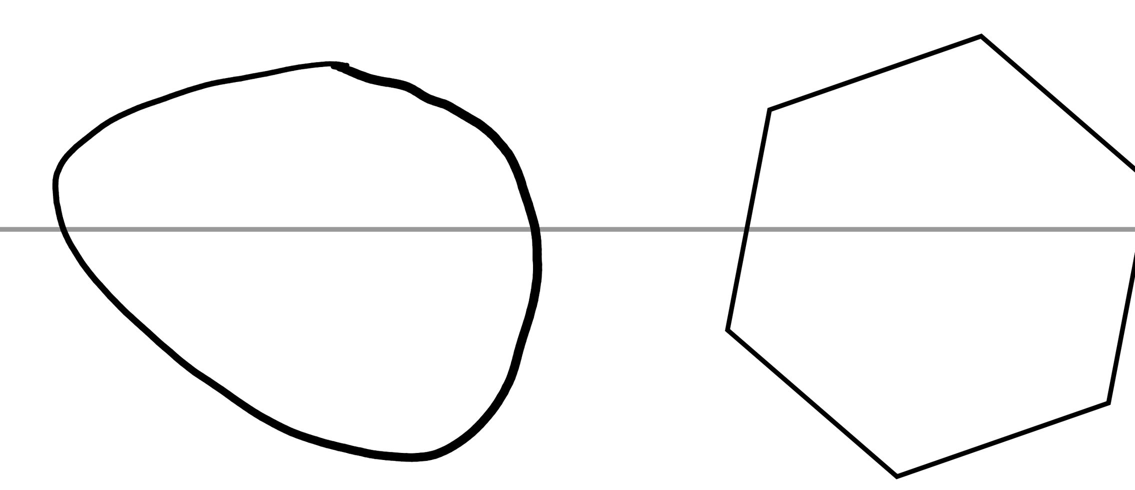
Lecture 9. Simplex method. dual problem.

Fundamental theorem of linear programming.

Consider the LP: max / min $c^T x$, s.t. $Ax \le b$, $x \ge 0$. Suppose the LP has ≥ 1 optimal solutions. Then $\exists$ an optimal solution at a vertex.

Vertex: $x \in$ polyhedron P is a vertex if $\exists\, n$ l.i. constraints tight at x.

Definition (extreme point): $x \in$ convex C is an extreme point if x is not a convex combination of other two points. i.e. $\nexists\, \theta, u \neq v \in C$, s.t. $x = \theta u + \bar{\theta} v$.



Proposition: Extreme points and vertices are equivalent in polyhedra.

Proof. "$\Leftarrow$". $x \in P$ is a vertex. $\Rightarrow \exists\, n$ linearly independent constraints tight at x. i.e. $\exists\, \tilde{A} \in \mathbb{R}^{n \times n}$ submatrix of $A \in \mathbb{R}^{m \times n}$, $\tilde{b} \in \mathbb{R}^n$ subsequence of $b \in \mathbb{R}^m$. s.t. $\tilde{A}$ invertible and $\tilde{A} x = \tilde{b}$. Suppose $x = \theta u + \bar{\theta} v$. Then $\theta \tilde{A} u + \bar{\theta} \tilde{A} v = \tilde{b}$. but $\tilde{A} u, \tilde{A} v \le b$. So $\tilde{A} u = \tilde{A} v = \tilde{b} \Rightarrow x = u = v$.

"$\Rightarrow$". Suppose an extreme point $x \in P$ is not a vertex. Let $I = \{ i : a_i^T x = b_i \}$. Then $\nexists\, n$ l.i. a_i's s.t. $i \in I$. So $\exists\, d \neq 0 \in \mathbb{R}^n$. $A_I d = 0$. Let $\begin{matrix} u = x + \varepsilon d \\ v = x - \varepsilon d \end{matrix}$.

We claim that $\exists\, \varepsilon > 0$ sufficiently small. s.t. $u, v \in P$. contradiction.

Note that $\forall i \in I$. $a_i^T u = a_i^T v = a_i^T x = b_i$. since $\forall i \in I$, $a_i^T d = 0$.

$\forall i \notin I$. let $f_i(w) = b_i - a_i^T w$ continuous. and $f(x) > 0$. $\Rightarrow \exists \varepsilon_i > 0$.

s.t. $f(u) \geq 0$. $f(v) \geq 0$. Then $\varepsilon = \min\{\varepsilon_i : i \notin I\} > 0$ is desired. □

Lemma. Extreme points exist iff P does not contain a line. ($P \neq \phi$)

Proof. Let line $l = \{y = x + tv : t \in \mathbb{R}\}$ for some fixed $x, v \in \mathbb{R}^n \in P$.

Clearly $\forall u \in l$ is not an extreme points. For any $u \notin l$ but $u \in P$.

claim that $u+v$, $u-v \in P$. so contradicts. Note that P is convex. so

$\forall \theta \in (0,1)$. $t \in \mathbb{R}$. $\theta u + \bar{\theta}(x + tv) \in P$. Let $t = 1/\bar{\theta}$. Then we have

$\theta u + \bar{\theta} x + v \in P$. Since P is also closed. $\lim\limits_{\theta \to 1} \theta u + \bar{\theta} x + v = u + v \in P$.

Conversely. if P does not have an extreme point. then no vertex. so

$\operatorname{rank}(A) < n$. $\exists d \neq 0 \in \mathbb{R}^n$. s.t. $Ad = 0$. $\forall x \in P$. $x + td \in P$. □

Proof of the fundamental theorem of linear programming:

Let $P = \Omega$ be the feasible set. $Q = \Omega_{opt} \neq \phi$. $x \geq 0 \Rightarrow P$ contains no

lines $\Rightarrow Q \subseteq P$ contains no lines $\Rightarrow Q \neq \phi$ has an extreme point x^*.

We now show that x^* is also extreme in P. Suppose not, $\exists x_1, x_2 \in P$.

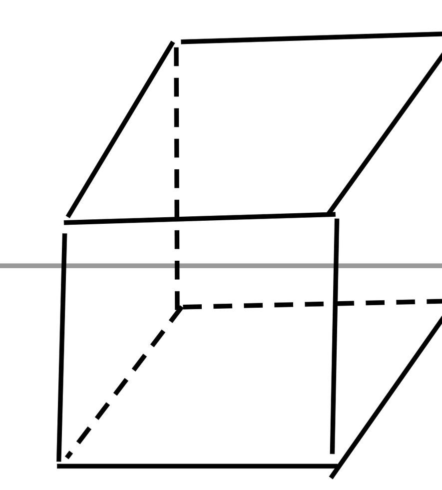
$\theta \in (0,1)$. s.t. $x^* = \theta x_1 + \bar{\theta} x_2$. Let $v = c^T x^* = \theta c^T x_1 + \bar{\theta} c^T x_2$.

Since $x^* \in Q$ is an optimal solution of P. $c^T x_1, c^T x_2 \le v$. Thus

$c^T x_1 = c^T x_2 = v \Rightarrow x_1, x_2 \in Q$, contradicts x^* extreme in Q. $\square$

Algorithm: find optimal solutions by enumerating $\le \binom{m}{n}$ vertices of P.

However, consider an n-dimensional $[0,1]^n$ cube. only $2n$ inequalities. but 2^n vertices.



Simplex method: start from a vertex and move to a better neighbor.

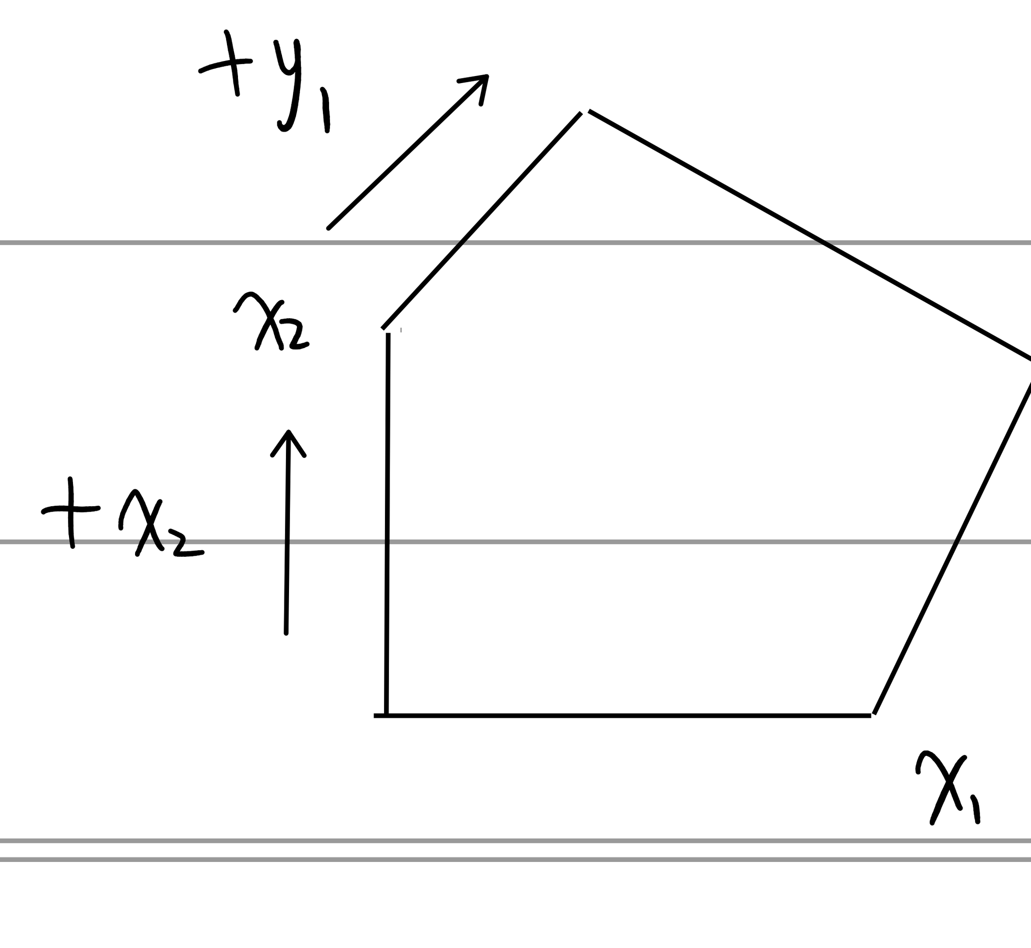
Definition: two vertices are neighbors if they share $n-1$ tight constraints.

In general computing neighbors is also hard. but it is easy for 0.

1. For convenience. suppose 0 is feasible. then is a vertex. start at 0.
2. Consider the goal: $\min c^T x$. if $\forall c_i < 0$. then should increase x_i
3. Increase x_i to make another constraint. say $a_j^T x = b_j$, tight.
4. Shift coordinates. Let $y_i = b_j - a_i^T x$. Replace x_i by y_i.
5. Check if $\forall c_i \ge 0$. If so we are done. otherwise go to 3. if $c_i < 0$.

Example. $\min -2x_1 - 5x_2$

subject to
$$2x_1 - x_2 \le 4$$
$$x_1 + 2x_2 \le 9$$
$$-x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0.$$



initial LP

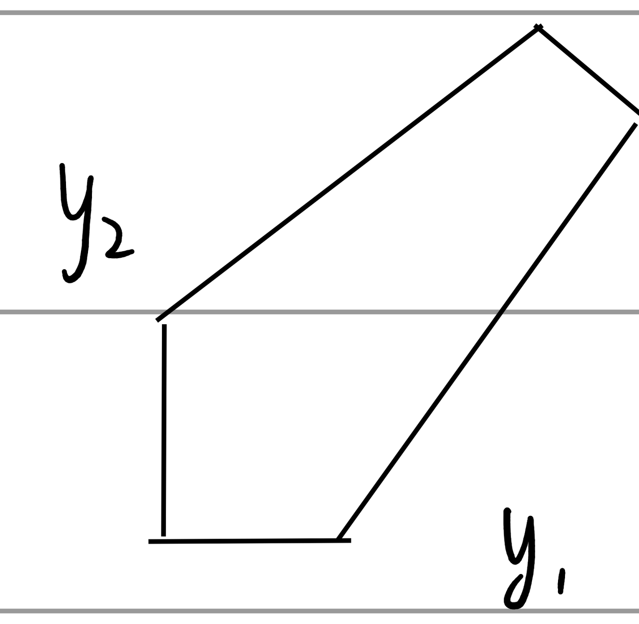
start from $(0,0)$

increase x_2 to make $-x_1+x_2 \le 3$ tight. let $y_1 = x_1$. $y_2 = 3+x_1-x_2$.

rewrite: $\min -15-7y_1+5y_2$

s.t. $y_1+y_2 \le 7$ $\quad -y_1+y_2 \le 3$.

$3y_1-2y_2 \le 3$. $\quad y_1\ y_2 \ge 0$.



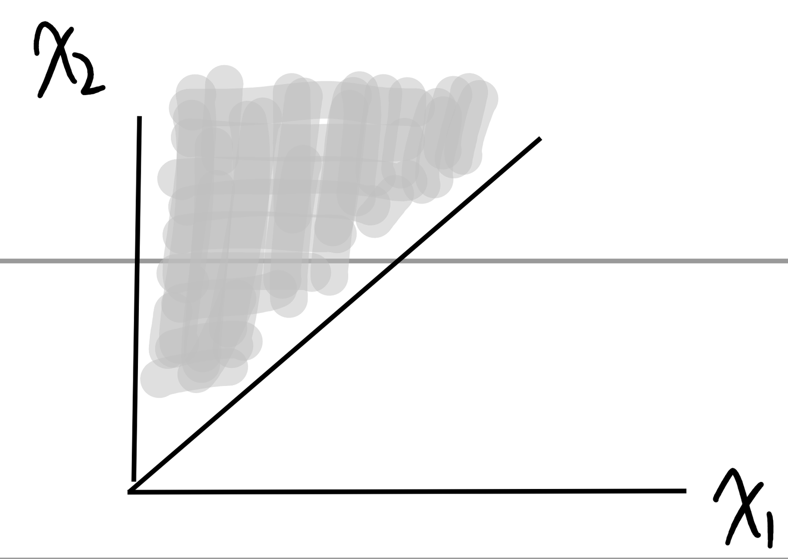
increase y_1 to make $3y_1-2y_2 \le 3$ tight. let $z_1 = 3-3y_1+2y_2$. $z_2 = y_2$

rewrite: $\min -22+\frac{7}{3}z_1+\frac{1}{3}z_2$. s.t. --- $z_1, z_2 \ge 0$. done!

Degeneracy 退化. more than n constraints tight at a vertex.

$x_1 - x_2 \le 0$

$x_1, x_2 \ge 0$.



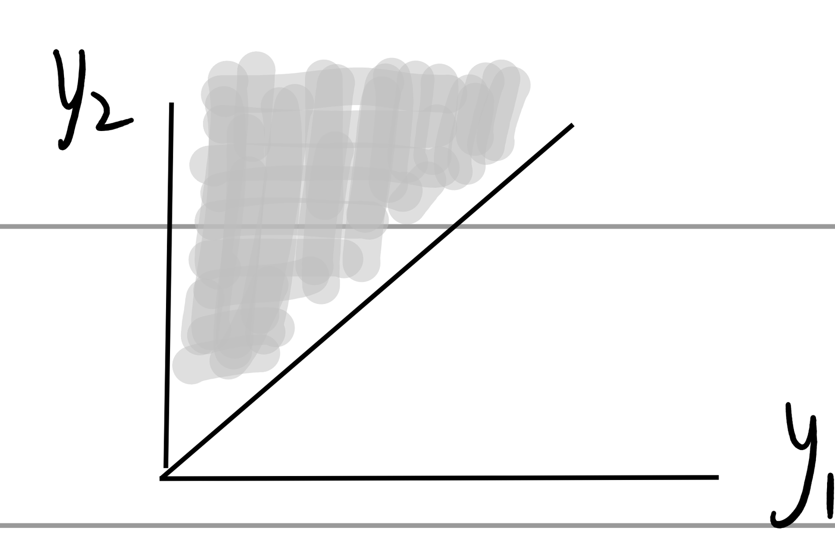
increase x_1

make $x_1-x_2 \le 0$ tight. $x_1=0$.

$y_1 = x_2-x_1$. $y_2 = x_2$

$y_1 \ge 0 \quad y_2 \ge 0$

$y_1 - y_2 \le 0$. increase y_1



fail!!!

how to fix degeneracy? break cycles / add perturbation $b_i' = b_i \pm \varepsilon_i$

Another question: how to use simplex method if origin is not feasible?

If $x=d$ is known to be feasible. let $y_i = d_i - x_i$. rewrite LP so that $y=0$ is feasible. However, y_i may <0. Let $y_i = y_i^+ - y_i^-$.

If none feasible solution is known. apply the "two-phase" simplex method.

consider a LP: $\min c^T x$. subject to $Ax=b$, $x \ge 0$. assume $b \ge 0$.

add slack variables $s_1 \dots s_m$ s.t. $Ax+s=b$. $x, s \ge 0$.