Lecture 10. LP duality.

Recall simplex method: move from a vertex to a better neighbor.

two vertices are neighbors if they share $n-1$ tight constraints

If we are at origin. we know it is a vertex since $\forall i$. $x_i \geq 0$ are tight.

So there are n neighbors of origin. each of them has $n-1$ coordinates 0.

How to check "better"? $\min \sum c_i x_i$ $\quad x_i > 0$ better iff $c_i < 0$.

choose i that $c_i < 0$. increase x_i until some constraint $a_j^T x \leq b_j$ tight.

Now there are n constraints tight: $\begin{cases} x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n = 0 \\ a_j^T x = b_j \end{cases}$

Shift coordinates so that $(0, 0, \ldots 0, x_i = r, 0, \ldots, 0)$ become origin.

Let $y_1 = x_1$. $y_2 = x_2$, $\ldots$ $y_{i-1} = x_{i-1}$. $y_i = b_j - a_j^T x$, $y_{i+1} = x_{i+1}$, $\ldots$ $y_n = x_n$.

Simplex method is not guaranteed to be poly-time. However, fortunately. the smoothed analysis, introduced by Teng-Spielman, shows that simplex method runs in poly-time in average if we add Gaussian perturbation.

If origin is not feasible. suppose $x = d$ feasible. let $y = d - x = y^+ - y^-$.

One more question: if feasible solutions are unknown?

consider a LP: $\min c^T x$, subject to $Ax = b$, $x \geq 0$. assume $b \geq 0$.

add slack variables $s_1 \ldots s_m$ s.t. $Ax + s = b$. $x, s \geq 0$.

Clearly $\exists$ a trivial solution $x = 0$, $s = b$. check if $s = 0$ possible

Solve LP: $\min s_1 + s_2 + \cdots + s_m$, s.t. $Ax + s = b$, $x, s \geq 0$.

Dual problem of a linear program.

Now consider at the end of the simplex method: $\min c^T x$. $c \geq 0$.

$\min 3x_1 + 2x_2$. s.t. $x_1 + x_2 \leq 5$. $x_1 \leq 3$. $x_1, x_2 \geq 0$.

Trivially $x_1 = x_2 = 0$ is optimal. How about $\max 3x_1 + 2x_2$?

Claim: $3x_1 + 2x_2 \leq 13$ since $3x_1 + 2x_2 = 2(x_1 + x_2) + x_1$.

In general. given $A = (a_{ij}) \in \mathbb{R}^{m \times n}$. $b \in \mathbb{R}^m$. assign y to constraints.

$$
\begin{array}{llr}
\max & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n. & \\
\text{s.t.} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1. & y_1 \\
 & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2. & y_2 \\
 & \vdots \qquad\qquad\qquad\quad \vdots & \vdots \\
 & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m & y_m.
\end{array}
$$

if $y_j \geq 0$, and $\sum y_i a_{ij} \geq c_j$ for all j. $\sum y_i b_i$ is an upper bound.

$\Rightarrow \max c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \leq \min y_1 b_1 + y_2 b_2 + \cdots + y_m b_m$.

Construct the LP. $\min y^T b$. s.t. $y^T A \geq c^T$. $y \geq 0$. dual LP.

Proposition: The dual of the dual is the primal.

Proof: Rewrite dual: $\max -b^T y$. s.t. $-A^T y \le -c$. $y \ge 0$.

dual of dual is: $\min -z^T c$, s.t. $-z^T A \ge -b^T$. $z \ge 0$. $\square$.

Remark. If primal $a_i^T x \ge b_i$. then $y_i \le 0$. $a_i^T x = b_i$. then $y_i \in \mathbb{R}$.

Weak duality theorem

primal feasible $\longleftarrow$ - - - - - $\longrightarrow$ dual feasible.

If x is feasible for primal. y is feasible for dual. then $c^T x \le y^T b$

Proof. $c^T x \le y^T A x \le y^T b$. $(x \ge 0.\ y \ge 0)$. $\square$

Corollary. If $c^T x$ unbounded. then dual infeasible. (note $\min \phi = \infty$).

primal / dual	unbounded	infeasible	$\exists$ optimal
unbounded	X	✓	X
infeasible.	✓	? ? ✓	? X
$\exists$ optimal.	X	? X	? ✓

Strong duality theorem.

primal feasible $\longleftarrow\bullet\longrightarrow$ dual feasible.

If primal has finite optimal x^*. so is dual. and $c^T x^* = {y^*}^T b$.

Example of both infeasible: $\max\ 2x_1 - x_2$. s.t. $x_1 - x_2 \le 1$. $x_2 - x_1 \le -2$. $x_1, x_2 \ge 0$
$\min\ y_1 - 2y_2$ s.t. $y_1 - y_2 \le 1$. $y_2 - y_1 \le -2$. $y_1, y_2 \ge 0$.

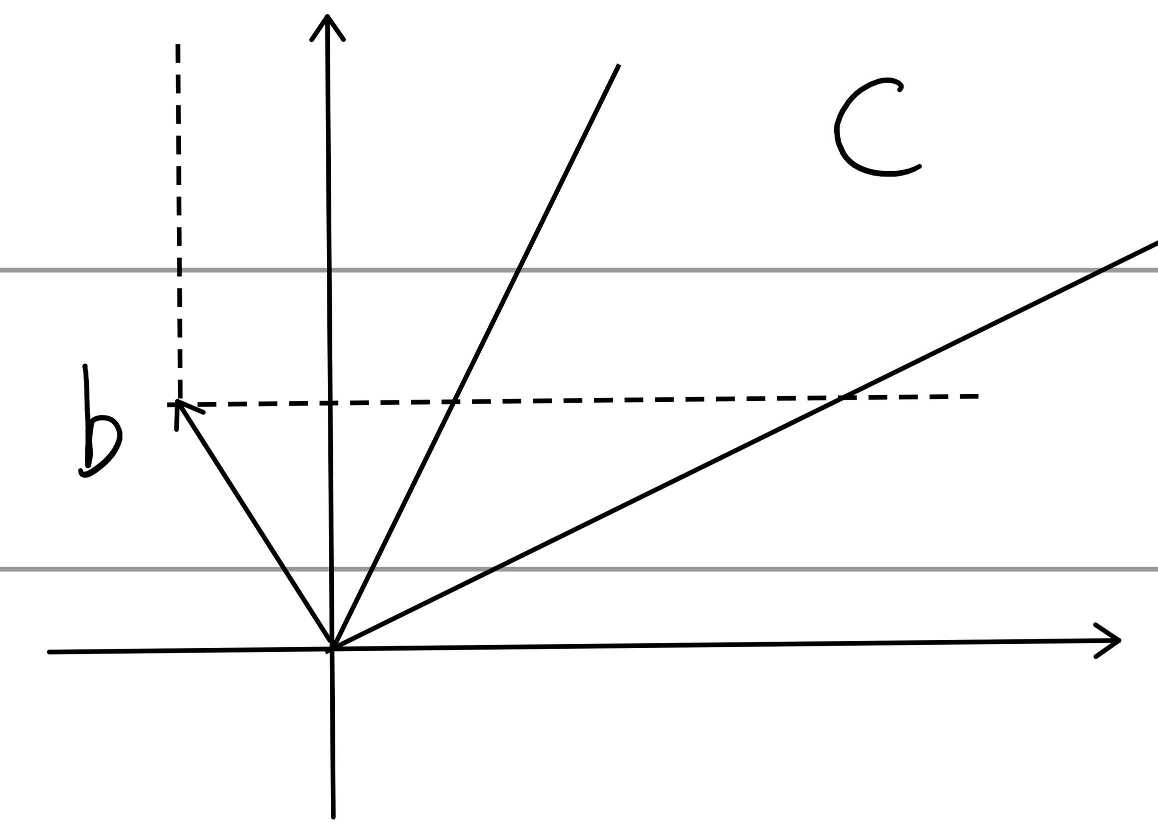
Proof of strong duality: application of Farkas' lemma.

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. then exactly one of the followings is true.

① $\exists\, x \in \mathbb{R}^n$. s.t. $Ax = b$. $x \ge 0$; ② $\exists\, y \in \mathbb{R}^m$. s.t. $A^T y \ge 0$. $b^T y < 0$.

We first consider the following corollary of Farkas' lemma:

exactly one of the followings is true.



① $\exists x \in \mathbb{R}^n$ s.t. $Ax \geq b$, $x \geq 0$.

② $\exists y \in \mathbb{R}^m$ s.t. $A^T y \geq 0$, $b^T y < 0$, $y \leq 0$.

proof of the corollary. Let $A' = (A, -I) \in \mathbb{R}^{m \times (m+n)}$. Apply FL.

① $\exists x' \in \mathbb{R}^{m+n}_{\geq 0}$ s.t. $A'x' = b \iff \exists x \in \mathbb{R}^n_{\geq 0}$ s.t. $Ax \geq b$.

② $\exists y \in \mathbb{R}^m$ s.t. $A'^T y \geq 0$, $b^T y < 0 \iff A^T y \geq 0$, $y \leq 0$, $b^T y < 0$.

Back to strong duality. Wlog. assume dual has an optimal solution y^*

Suppose strong duality is not true. Then $\nexists$ feasible x s.t. $c^T x = y^{*T} b$.

Let $\gamma = y^{*T} b$. It is equivalent to $\nexists x \in \mathbb{R}^n_{\geq 0}$ s.t. $\begin{pmatrix} -A \\ c^T \end{pmatrix} x \geq \begin{pmatrix} -b \\ \gamma \end{pmatrix}$

Applying Farkas' lemma, it gives that $\exists y \in \mathbb{R}^m_{\leq 0}$ and $w \in \mathbb{R}_{\leq 0}$

s.t. $\begin{pmatrix} -A \\ c^T \end{pmatrix}^T \begin{pmatrix} y \\ w \end{pmatrix} \geq 0$ and $\begin{pmatrix} -b \\ \gamma \end{pmatrix}^T \begin{pmatrix} y \\ w \end{pmatrix} < 0$.

Case 1. $w = 0$ Then $-A^T y \geq 0$ and $-b^T y < 0$, and $y \leq 0$.

Now $A^T(y^* - y) \geq A^T y^* \geq c$, $y^* - y \geq 0$, but $b^T(y^* - y) > \gamma$.

Case 2. $w < 0$, dividing w on both sides. then

$$(-A, c) \begin{pmatrix} y/w \\ 1 \end{pmatrix} \leq 0 \quad \text{and} \quad (-b^T, \gamma) \begin{pmatrix} y/w \\ 1 \end{pmatrix} > 0.$$

$\Rightarrow A^T(y/w) \geq c$ and $b^T(y/w) < \gamma$. $\Rightarrow y^*$ not optimal. $\square$

Corollary (complementary slackness). Suppose x, y feasible for (P)(D).

Then x, y are optimal iff $y^T(b - Ax) = 0$, $x^T(A^T y - c) = 0$.

Proof. $c^T x \le y^T A x \le y^T b$. $\begin{cases} \text{either } y_i = 0, \text{ or } (Ax)_i = b_i \text{ tight} \\ \text{either } x_j = 0, \text{ or } (A^T y)_j = c_j \text{ tight} \end{cases}$ $\square$

Applications of LP duality: zero-sum games and minimax theorem.

zero-sum games: rock-scissors-paper game.

	R	S	P
R	0	1	-1
S	-1	0	1
P	1	-1	0

$G \in \mathbb{R}^{n \times n}$: payoff matrix.

$x, y \in \mathbb{R}^n$: strategy distribution over $\{R, S, P\}$.

expected payoff: $\mathbb{E}[\text{payoff}] = \sum_{i,j} G_{ij} x_i y_j = x^T G y$

$x = y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $\mathbb{E}[\text{payoff}] = 0$. equilibrium. 均衡.

$x = (0, 0, 1)$. $y = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. $\mathbb{E}[\text{payoff}] = \frac{1}{4}$.

$x = (0, 0, 1)$. $y = (0, 1, 0)$ $\mathbb{E}[\text{payoff}] = -1$.

Player X: for fixed x. player Y's best strategy is to minimize

$\sum_{i,j} G_{ij} x_i y_j \Rightarrow$ X's goal is $\max_x \min_y \sum_{i,j} G_{ij} x_i y_j$.

Player Y: for fixed y. player X's best strategy is to maximize

$\sum_{i,j} G_{ij} x_i y_j \Rightarrow$ Y's goal is $\min_y \max_x \sum_{i,j} G_{ij} x_i y_j$.

We claim they are dual problems. For example. $G = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

If X choose strategy $x = (x_1, x_2)$. payoff of Y is $\begin{pmatrix} 3x_1 - 2x_2 \\ -x_1 + x_2 \end{pmatrix}$

goal of X: $\max\limits_{x_1+x_2=1} \min \{3x_1 - 2x_2, -x_1 + 2x_2\}$.

$\Leftrightarrow \max z$. s.t. $\begin{matrix} 3x_1 - 2x_2 \geq z & x_1 + x_2 = 1 \\ -x_1 + x_2 \geq z & x_1, x_2 \geq 0. \end{matrix}$

If Y choose strategy $y = (y_1, y_2)$. payoff of X is $\begin{pmatrix} 3y_1 - y_2 \\ -2y_1 + y_2 \end{pmatrix}$.

goal of Y: $\min\limits_{y_1+y_2=1} \max \{3y_1 - y_2, -2y_1 + y_2\}$.

$\Leftrightarrow \min w$. s.t. $\begin{matrix} 3y_1 - y_2 \leq w & y_1 + y_2 = 1 \\ -2y_1 + y_2 \leq w & y_1, y_2 \geq 0. \end{matrix}$

Minmax is the dual of maxmin. so equality holds by SD.

Theorem (von Neumann's Minimax Theorem).

$$\max_x \min_y x^T G y = \min_y \max_x x^T G y.$$

Theorem (Yao's min-max theorem).

$$\max_{x \in X} \mathbb{E}_A [c(A, x)] \geq \min_{a \in A} \mathbb{E}_X [c(a, X)].$$