CS-3334: Advanced Combinatorics **Fall 2022**

Lecture 1: Introduction, Pigeonhole Principle

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1.1 Introduction

1.1.1 What is Combinatorics?

Combinatorics is an area of mathematics concerned with properties of discrete structures, including counting, existence, optimality, etc. For example, combinatorics contains following subfields:

- enumerative and analytic combinatorics, which focus on counting the number of certain combinatorial objects;
- extremal and probabilistic combinatorics, which focus on how large or how small a combinatorial object can be, if it has some certain properties;
- graph theory, which studies graph, an important type of combinatorial structures; here are some interesting topics in graph theory that will not be covered in this course:
  - structural graph theory

    Neil Robertson and Paul Seymour proved the graph minor theorem as follows.

    Theorem 1.1 (Robertson–Seymour Theorem)

    For any minor closed property P, $\exists$ a finite set $\{H_1, \ldots, H_m\}$ such that $\forall G$, G satisfies $P \iff \nexists\, i$ such that H_i is a minor of G. ♡

    For instance, we know that K_5 and $K_{3,3}$ are not planar graphs. Wagner's theorem states that planar graphs can be defined as graphs without K_5 or $K_{3,3}$ as its minor. A minor of a given graph is another graph formed by deleting vertices, deleting edges, and contracting edges. When an edge is contracted, its two endpoints are merged to form a single vertex.
  - spectral graph theory
- $\cdots$

In this course, we will mainly concentrate on extremal combinatorics, and probabilistic and linear algebra tools.

1.1.2 Some interesting problems

- one problem from final exam last semester

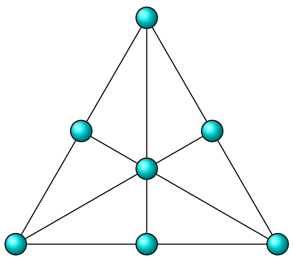
  Problem 1.1 For any graph G, define $f(G)$ as the number of cliques of size 3 plus the number of independent sets of size 3. For any 6-regular graph of size 30, what's the maximal (minimal) value of G?

  Answer: Consider the number of triples (u, v, w) such that u and v are adjacent, while u and w are not. As graph G is a 6-regular graph of size 30, there are $30 \times 6 \times 23$ such triples in total. For any three vertices that do not form a clique or an independent set, there exists exactly two such triples. Therefore, $f(G)$ equals to $\binom{30}{3} - \frac{1}{2} \times 30 \times 6 \times 23$. ■

  Extensions: How about non-regular graphs?
- unit distance problem (distinct distance problem)
- incidence geometry

  Problem 1.2 Does there exist n points and m lines such that each line passes through 3 points?

  When $n = 7$ and $m = 6$, **Fano plane** is a solution.



- graph drawings and crossing number

  Problem 1.3 What is the least number of crossing points if we draw a $K_{3,3}$ on a plane?
- error-detecting/correcting codes
  - Identity Card number
  - Madhu Sudan's list-decoding algorithm

1.2 Ramsey Number

1.2.1 Definition of Ramsey Number

We start from two simple problems as follows.

Problem 1.4 Prove that for any 2-coloring of edges of K_6, there exists a yellow K_3 or a blue K_3.

Proof Consider a vertex u in K_6, there are 5 edges connecting u, which implies that there exists at least 3 monochromatic edges connecting u. Without loss of generality, assume that there are 3 yellow edges connecting u and a, b, c, respectively. Now, consider the three edges between a, b, c. If all of them are blue, then they form a blue K_3. Otherwise, assume that the edge between a and b is yellow, then u, a, b form a yellow K_3, which completes the proof.

It is easy to find a 2-coloring of edges of K_5 such that the conclusion doesn not hold.

Problem 1.5 Prove that for any 2-coloring of edges of K_{10}, there exists a yellow K_3 or a blue K_4.

Proof Consider a vertex u in K_{10}. Among the 9 edges connecting u, there exists 4 yellow ones or 6 blue ones.

If there are 4 yellow edges connecting u and a, b, c, d, respectively. Consider the edges between a, b, c, d. If all of them are blue, then they form a blue K_4. Otherwise, assume that the edge between a and b is yellow, then u, a, b form a yellow K_3.

If there are 6 blue edges connecting u and other 6 distinct vertices. Consider the induced subgraph of these 6 vertices. According to Problem 1.4, there exists a yellow K_3 or a blue K_3. If a yellow K_3 exists, then the proof is completed. If a blue K_3 exists, then these 3 vertices and u form a blue K_4.

Notice that the same conclusion holds for any 2-coloring of edges of K_9. This is because we can always find a vertex u that there exists 4 yellow edges or 6 blue edges connecting it.

Now, we introduce the definition of **Ramsey Number** as follows.

Definition 1.1 (Ramsey Number)

$R(s, t)$ is defined as the smallest n satisfying: Given K_n, for any 2-coloring of edges of K_n, either a yellow K_s or a blue K_t exists. ♣

From above, we already know that $R(3, 3) = 6$ and $R(3, 4) \leq 9$. We can also find that $R(s, t) \leq R(s-1, t) + R(s, t-1)$ of which the proof is similar to that in Problem 1.5. We will give an upper bound for $R(s, t)$ in the next lecture.

The key to the above proofs is the **Pigeonhole Principle**.

Theorem 1.2 (Pigeonhole Principle)

Let N, R be two finite sets of size $|N| = n > r = |R|$. Consider a mapping $f : N \to R$ and non-negative integers $a_1, a_2, \ldots, a_r$ such that $\sum_{i=1}^{r} a_i < n$. Then, there exists $s \in R$ such that $|f^{-1}(s)| \geq a_s + 1$. ♡

1.2.2 Erdős-Szekeres Theorem (Happy ending problem)

The “**happy ending problem**" is the following statement.

Theorem 1.3 (Esther Klein, 1933)

Any five points in a plane in general position has a subset that forms a convex quadrilateral. General position means that no two points coincide and no three points are collinear. ♡

In 1935, Paul Erdős and George Szekeres proved the following generalisation:

Theorem 1.4 (Paul Erdős & George Szekeres, 1935)

For any positive n, any sufficiently large finite set of points in general position has a subset of n that forms a convex polygon. ♡

Remark. It is a fundamental theorem of combinatorial geometry. Four years later (1937), Esther Klein became Esther Szekeres. (That's why Erdős name it the "happy ending problem"!) During World War II, George and Esther escaped to China and lived in Hongkew, Shanghai. They moved to Australia after the war.

Before giving the proof of this theorem, let's first see another theorem proved by Paul Erdős and George Szekeres at the same time.

Theorem 1.5 (Paul Erdős & George Szekeres, 1935)

Any sequence of length $mn+1$ with distinct numbers has an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$. ♡

Proof Define a_i, b_i as the length of the longest increasing, decreasing subsequence that ends at the i-th number, respectively. For any $i < j$, $a_i \neq a_j$ or $b_i \neq b_j$ holds. (This is because if the i-th number is smaller than the j-th one, then $a_i < a_j$. Otherwise, $b_i < b_j$.)

If the longest increasing subsequence has length at most n and the longest decreasing subsequence has length at most m, then there must exist $i < j$ such that $a_i = a_j$ and $b_i = b_j$ due to the **Pigeonhole Principle**, which leads to the contradiction.

Now, let's introduce the proof of Theorem 1.4.

Proof Let's prove that for any $\binom{p+q}{p} + 1$ points in general position, there exists a concave polyline of length $p+1$ or a convex polyline of length $q+1$. (Note that a concave/convex polyline will lead to a convex polygon. If we have proved this, then set $p = q = n - 2$ and the whole proof will be completed.)

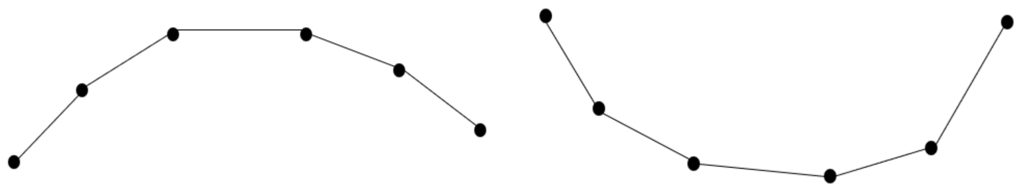


Figure 1.1: The left picture shows a concave polyline of length 5, while the right one shows a convex polyline of length 5.

We will finish the proof by induction on p and q. It obviously holds when $p = 1$ or $q = 1$. Suppose there are $\binom{p+q}{p} + 1$ points in general position and no convex polyline of length $q+1$ exists. By induction hypothesis, a concave polyline of length p exists as $\binom{p+q}{p} + 1 \geq \binom{p+q-1}{p-1} + 1$. Remove the rightmost point of the concave polyline and add the point into a set S. Repeat the process for $\binom{p+q}{p} + 1 - \binom{p+q-1}{p-1} = \binom{p+q-1}{p} + 1$ times. Based on the induction hypothesis, there exists either a concave polyline of length $p+1$, or a convex polyline of length q in S. If there exists a concave polyline of length $p+1$, then we're done. Otherwise there exists a convex polyline of length q in S. In this way, we can find $p+q+1$ points such that the left $p+1$ points form a concave polyline while the right $q+1$ points form a convex polyline. It's easy to show that either the left $p+2$ points form a concave polyline of length $p+1$, or the right $q+2$ points form a convex polyline of length $q+1$, which completes the proof.

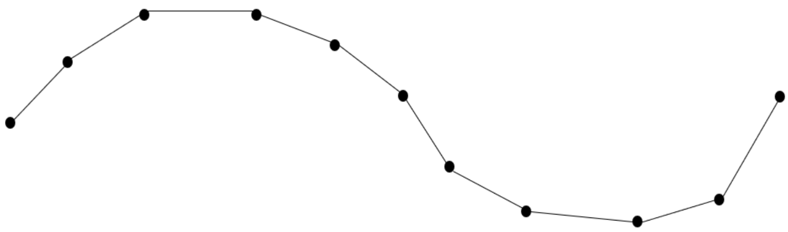


Figure 1.2: The picture shows the case that 11 points form a polyline of length 10, where $p = q = 5$. It is easy to see that left 7 points form a concave polyline of length $p + 1 = 6$.

1.2.3 Generalization and Applications of Ramsey Problem

In this section, we introduce generalized **Ramsey theorem**.

Theorem 1.6 (Frank Ramsey, 1930)

Let $r \geq 1$ and $q_i \geq r$ for $1 \leq i \leq s$. There exists a minimal integer $N = R(q_1, \ldots, q_s; r)$ such that for any coloring $f : E(K_N^{(r)}) \to [s]$ of edges of the complete r-uniform hypergraph $K_N^{(r)}$, $\exists i \in [s]$ and a copy of $K_{q_i}^{(r)}$ of color i. ♡

Denote $R(\overbrace{q, \ldots, q}^{s \text{ times}}; r)$ by $R_s(q; r)$.

With this theorem, we can prove **Schur's theorem**.

Theorem 1.7 (Issai Schur, 1916)

Given any $c > 0$, there exists $S(c)$ such that no matter how we color $[S(c)]$ with c colors, there exists monochromatic x, y, z that $x + y = z$. ♡

Proof Take $S(c) = R_c(3; 2)$. For any edge (i, j) in graph $K_{S(c)}$, color it by $|i - j|$'s color. According to Theorem 1.6, there exists a monochromatic K_3 in graph $K_{S(c)}$. Assume that $u < v < w$ form a monochromatic K_3. Set $x = v - u, y = w - v, z = w - u$. Obviously, x, y, z have the same color and $x + y = z$, which completes the proof.

Also, we obtain an easier proof of Theorem 1.4.

Proof $N = R(n, 5; 4)$ points suffice. For any four vertices, if they form a convex quadrilateral, use the first color (let's assume it's blue) to color the corresponding hyperedge. Otherwise, use the second color (let's assume it's yellow). According to Theorem 1.6, there exists a blue $K_n^{(4)}$ or a yellow $K_5^{(4)}$. However, a yellow $K_5^{(4)}$ can never exist based on Theorem 1.3, which implies that there exists n points where any 4 of them form a convex quadrilateral. It further yields that these n points form a convex polygon (why?).