Lecture 1. Introduction. Pigeon-hole principle.

Course info.

Office hours: after class or Thurs. afternoon.

Materials: Extremal Combinatorics.

Yufei Zhao's notes (both probabilistic & polynomial).

Probabilistic method. Noga Alon. Joel Spencer

Polynomial method Larry Guth.

Syllabus.

Pigeonhole principle. Ramsey. Schur. van der Warden

additive combinatorics. Green-Tao. happy ending.

Intersecting set families. sunflower. Erdős-Ko-Rado.

Probabilistic method. expectation moment method.

Lovász local lemma. concentration. regularity lemma?

Entropy container. random graphs.

Linear algebra method. odd town / even town.

polynomial. Combinatorial Nullstellensatz. codes.

Interesting problems

1. final exam.

   # of $\triangle$'s and $\therefore$'s. on regular graphs.

   how about non-regular graphs?

   # of edges in unit distance graphs.

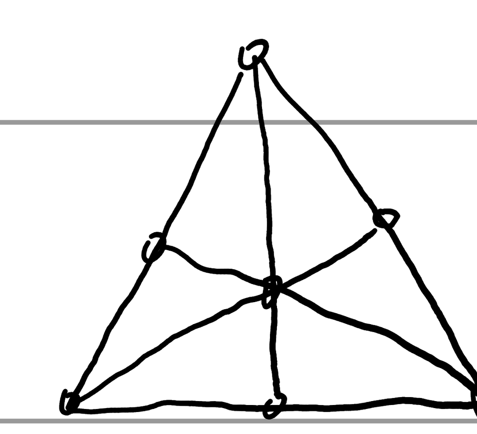
   distinct distance problem.

2. geometry.

   graph drawings. crossing number.

   incidence geometry.

   7 trees. 6 lines. 3 on each line.



Fano plane.

3. odd town / even town.

4. graph coloring. (large χ and girth).

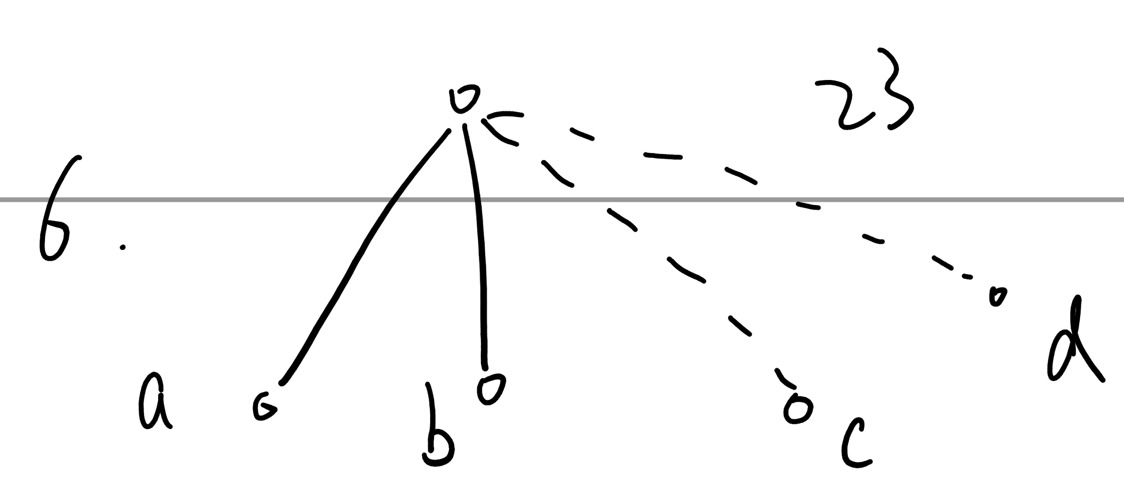
5. graph decomposition.

6. error-detecting and -correcting codes.

7. happy ending problem.

8. June Huh. log-concavity chromatic polynomial.

Consider a 6-regular graph of size 30.



count # of $\triangle$ and $\triangle$

assume we want max.

However, we do not know adjacency of $abcd$.

if we want max. $a \sim b$ $c \not\sim d$. globally inconsistent.

consider # of $\triangle$'s and $\triangle$'s.

$\{a, b\} \times \{c, d\}$ counts no matter whether they're adjacent. each of $\triangle$ and $\triangle$ count twice.

of $\triangle$ and $\triangle$ $= 30 \times 6 \times 23 / 2$.

A natural question: is it possible # of $\triangle$ and $\triangle$ $= 0$?

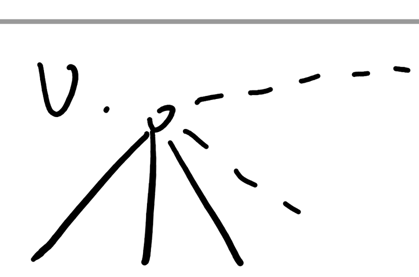
r-regular graph of n-vertex: $\binom{n}{3} = n \cdot r \cdot (n-1-r)/2$

$\Rightarrow (n-1)(n-2) = 3r(n-1-r)$. $\Rightarrow \begin{cases} n=4 \\ r=1,2 \end{cases}$ or $\begin{cases} n=5 \\ r=2 \end{cases}$

$n \geq 6$. regular graphs always have $\triangle$ or $\triangle$.

The next question: how about non-regular graphs?

We claim: any graph of size ≥ 6 has $\triangle$ or $\triangle$.



$\deg(v) + \overline{\deg}(w) = 5$. w.l.o.g $\deg(v) \geq 3$. done.

Ramsey problem.

Given a complete graph K_n. assign each edge a color (B / Y)

For any fixed r, s, the following holds if n is sufficiently large:

no matter how we color the graph K_n, there exists a

monochromatic K_r or K_s (blue K_r or yellow K_s).

Proof. By induction. $R(r,s) :=$ the smallest n.

w.l.o.g assume $r \leq s$. Otherwise $R(r,s) = R(s,r)$.

If $r=2$. obviously $R(r,s) = s$.

If $r>2$. $R(r,s) \leq R(r-1,s) + R(r,s-1)$. $\square$.

Pigeon-hole Principle.

Let N. R be two finite set of size $|N| = n > r = |R|$.

Consider a mapping $f: N \to R$. Then $\exists\, s \in R$. $|f^{-1}(s)| \geq \frac{n}{r}$.

Numbers:

Take any $n+1$ numbers of $[2n]$. Then:

1. Two among them are relative prime;

2. Two among them satify that one divides the other.

Sums.

Given n integers $a_1, \cdots, a_n$, there exist consecutive numbers $a_{t+1}, a_{t+2}, \cdots, a_{t+l}$ whose sum $\sum_{k=t+1}^{t+l} a_k$ is a multiple of n.

Happy ending problem. (named by Paul Erdős).

Claim (Esther Klein, 1933).

Any five points has a subset that forms a convex quadrilateral.

Question (Esther Klein. 1933)

How about general convex n-polygon?

Theorem (Paul Erdős & George Szekeres, 1935).

For any positive n, any sufficiently large finite set of points has a subset of size n that forms a convex polygon.

Remark. It is a fundamental theorem of combinatorial geometry.

Four years later (1937). Esther Klein became Esther Szekeres.

During WWII. George and Esther escaped to China, and lived in Hongkew, Shanghai. They moved to Australia after the war.

Alternation. any $mn+1$ distinct numbers has LIS $n+1$ or LDS $m+1$.

Dilworth's theorem (Robert Dilworth, 1950). (dual: Mirsky).

The largest antichain has the same size as the smallest chain decomposition, for any finite partially ordered set.

Back to Erdös–Szekeres theorem. Proof by induction.

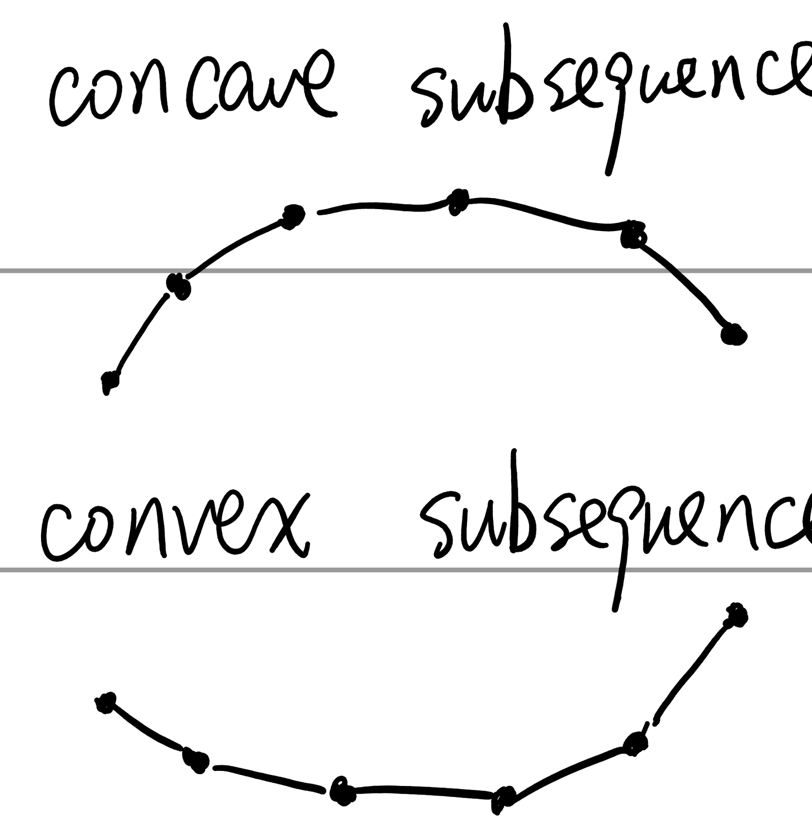
Let $n = N(k)$ be the smallest number of points in order to always form a convex polygon of size k. claim: $1+2^{k-2} \le N(k) \le 1 + \binom{2k-4}{k-2}$.

Lemma: $\binom{p+q}{p}+1$ points have $(p+1)$-concave or $(q+1)$-convex.

Proof. Assume no $(q+1)$-convex. by induction hypothesis. $\exists$ p-concave.

Remove the rightmost point. and add it into a set S.

Still $\exists$ $(p+1)$-concave. Repeat the process.

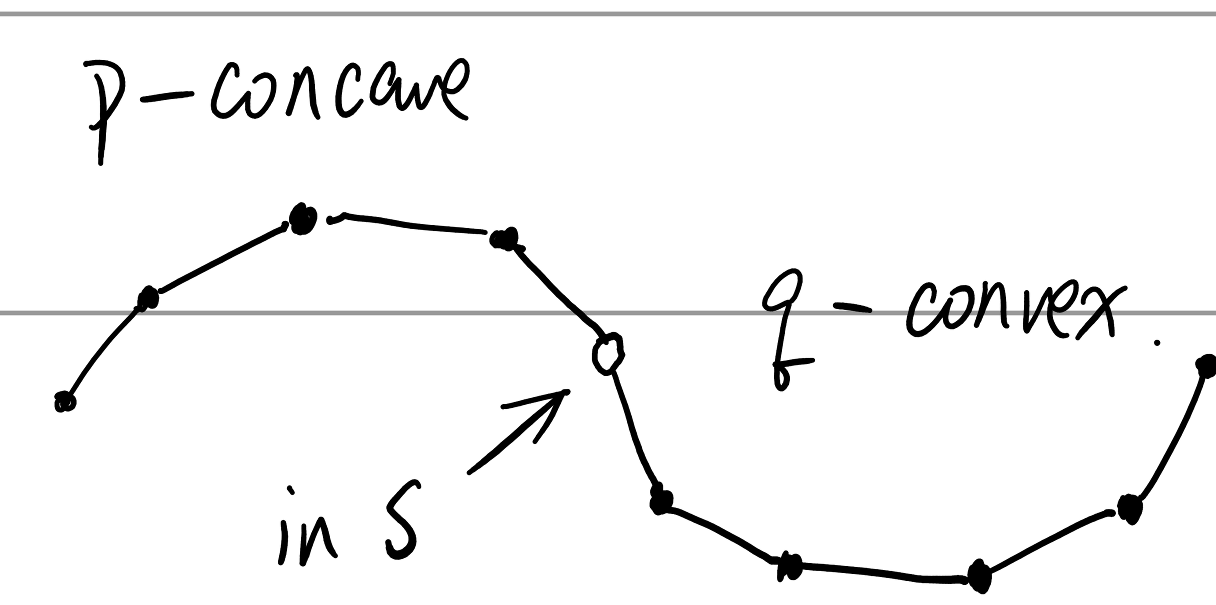


Then $|S| \geq \binom{p+q}{p}+1 - \binom{p+q-1}{p-1}+1$

$$= \binom{p+q-1}{p}.$$

So $\exists$ q-convex subsequence in S.

Then we have p-concave + q-convex :



$\square$,

Remark: $N(k) = 1+2^k$ conjectured. $N(4)=5$. $N(5)=9$. $N(6)=17$.

Still open in general. $N(k) \le 2^{k+o(k)}$ in JAMS, 2017.

Generalization and applications of Ramsey problem.

Schur's theorem (Issai Schur, 1916)

Given any $c>0$. $\exists S(c)$. s.t. no matter how we color $[S(c)]$ by c colors. $\exists x, y, z$ of the same color that $x+y=z$.

Proof. $N_c(3;2)$. edge (i,j) colored by $|i-j|$'s color. $\square$.

Question: Does $N_c(3;2)$ exist? Generalization of Ramsey.

Exercise: Show $N(3,3,3;2)=17$.

Theorem (Frank P. Ramsey, 1930). F. P. Ramsey 1903–1930

Let $r\geq 1$ and $q_i \geq r$ for $1\leq i\leq s$. There exists a minimal positive integer $N=N(q_1, q_2, \ldots, q_s; r)$ such that for any coloring $f: E(K_N^{(r)}) \to [s]$ of all edges of the complete r-uniform hypergraph $K_N^{(r)}$, $\exists\, i\in[s]$ and a copy of monochromatic $K_{q_i}^{(r)}$.

Notation: denote $N(q, q, \ldots, q; r)$ by $N_s(q; r)$.

Proof of Erdős–Szekeres theorem: $N=N(k,5;4)$ suffices.

$\forall$ 4 points. color the hyper edge by B if convex. and color the hyper edge by Y otherwise. Due to Ramsey, done. $\square$.

Theorem (Issai Schur, 1916)

Let $n \geq 1$. $\exists$ sufficiently large $s(n)$ s.t. $\forall$ prime $p > s(n)$.

$x^n + y^n \equiv z^n \pmod p$ has an integer solution in $[p-1]$.

Proof. $\mathbb{Z}_p^+ = [p-1] = \{g, g^2, \ldots, g^{p-1}\}$. (why?)

(Let $\psi(d) = \#$ of elements of order d. Then $\psi(d) \leq \phi(d)$

by Lagrange's theorem: n-degree polynomial has $\leq n$ roots

mod p, and $1, m, m^2, \ldots, m^{d-1}$ are roots of $x^d \equiv 1 \bmod p$.

Then $p-1 = \sum_{d \mid p-1} \psi(d) \leq \sum_{d \mid p-1} \phi(d) = p-1$. completes proof.)

$\left(\sum_{d \mid n} \phi(d) = \sum_{d \mid n} \phi\left(\frac{n}{d}\right) = \sum_{d \mid n} \sum_{\substack{t \leq n/d \\ \gcd(n/d, t) = 1}} 1 = \sum_{d \mid n} \sum_{\substack{t \leq n \\ \gcd(n,t) = d}} 1 = n. \right)$

color $g^{ns+r} \to r$. Then by Schur's theorem. $\exists k$ s.t.

$g^{nr+k} + g^{ns+k} = g^{nt+k}$. $p \nmid g^k \Rightarrow (g^r)^n + (g^s)^n = (g^t)^n$. $\square$

By the proof of Schur's theorem, it is easy to see:

Folkman's theorem (a memorial to Jon Folkman, 1938 - 1969).

For any positive integer c and r, $\exists n = n(c, r)$ s.t. no matter

how we color $[n]$ by c colors. $\exists x_1, \ldots, x_r \in [n]$ and $\sum x_i \leq n$

s.t. all $2^r - 1$ sums $\sum_{i \in I} x_i$, $\phi \neq I \subseteq [r]$ are of the same color.

Generalization of Schur's theorem: $x+y-z=0 \to x+y-2z=0$?

Theorem (van der Waerden, 1927).

For any c, l. $\exists\, W = W(c,l)$. s.t. any c-coloring of $[W]$ contains a monochromatic arithmetic progression of length l.

Szemerédi's theorem (Endre Szemerédi, 1975).

Any subset S of natural numbers with positive upper density (namely $\limsup_{n\to\infty} \frac{1}{n}|S\cap[n]| > 0$) contains infinitely many arithmetic progressions of length k, for all positive integer k.

Green-Tao theorem (Ben Green and Terence Tao, 2004).

The sequence of prime numbers contains arbitrarily long AP.

Question: $x+y-2z=0$ exists. how about $x+y-3z$?

Rado's theorem (Richard Rado, 1933).

For any equation $a_1x_1+\cdots+a_nx_n=0$, the followings are equivalent:

(1) $\forall c>0$, $\exists\, N=N(c)$ s.t. any c-coloring of $[N]$ contains a monochromatic solution to $a_1x_1+\cdots+a_nx_n=0$.

(2). $\exists$ a nontrivial $\{0,1\}$ solution to $a_1x_1+\cdots+a_nx_n=0$.