Lecture 3. Introduction to the probabilistic method.

Recall Erdős's proof of the lower bound of $R(k,k)$.

Color edges u.a.r. $\forall S \in \binom{[n]}{k}$. $\Pr[S \text{ induces mono-}K_k] = 2^{1-\binom{k}{2}}$

$\Rightarrow \Pr[\nexists \text{ mono-}K_k] > 1 - \binom{n}{k} 2^{1-\binom{k}{2}} > 0 \Rightarrow$ existence. $\square$

Review of probability. what do we mean when talk about probability.

- Throw a dice. two outcomes: 6 or not 6. $\Pr[6] = \frac{1}{2}$. ???

not even wrong.

- Uniformly pick a natural number. ???

- Bertrand paradox. $\Pr[\text{the length of a chord} \geq \text{radius}] = $???

Definition. Probability space. $(\Omega, \mathcal{F}, \Pr[\cdot])$.

- Ω is the set of "outcomes" (sample space. countable / uncountable)

- $\mathcal{F}$ is a σ-algebra (a set of all possible "events"). on which we can define probability. $\mathcal{F}$ is a σ-algebra if $\mathcal{F}$ satisfies:

  1. $\phi \in \mathcal{F}$; 2. $\forall A \in \mathcal{F}$. $A^c \in \mathcal{F}$; 3. $\forall A_1, \dots A_n, \dots \in \mathcal{F}$. $\cup A_i \in \mathcal{F}$.

- $\Pr[\cdot] : \mathcal{F} \to [0,1]$ is a function such that.

  1. $\Pr[\phi] = 0$. $\Pr[\Omega] = 1$; 2. $\forall$ disjoint $A_1 \dots A_n \dots \in \mathcal{F}$. $\Pr[\cup A_i] = \sum \Pr[A_i]$.

In a well-defined probability space $(\Omega, \mathcal{F}, \Pr)$. $\Pr[A] > 0 \Rightarrow A \neq \phi$.

Union bound: $\Pr[\cup A_i] \le \sum \Pr[A_i]$ (countable many).

Principle of inclusion and exclusion $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\phi \ne I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \Pr\left[\bigcap_{i\in I} A_i\right] = \sum_{\phi \ne I \subseteq [n]} (-1)^{|I|+1} \Pr\left[\bigcap_{i \in I} A_i\right]$$

Boole-Bonferroni inequality.

$$\sum_{k=1}^{2t} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \Pr[\cap A_i] \le \Pr[\cup A_i] \le \sum_{k=1}^{2t+1} (-1)^{k+1} \sum_{I\in\binom{[n]}{k}} \Pr[\cap A_i]$$

Conditional probability: $\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$.

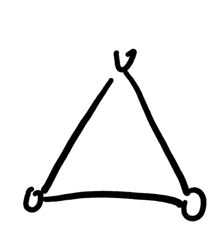
Roughly, a probability function is a weight function of each subset, and countably additive. In principle, the finite probability arguments can be rephrased as a counting proof, but usually complicated without probability.

2-colorable hypergraphs: lower bound / upper bound

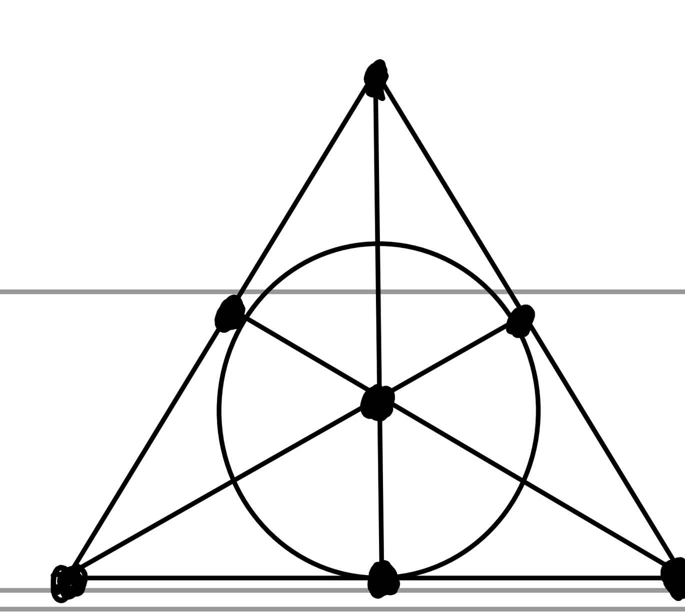
An k-uniform hypergraph $H = (V, E)$, where $E \subseteq \binom{V}{k}$, is c-colorable if V can be colored with c colors s.t. no edge is monochromatic.

2-colorable: property B. (Bernstein, 1908). $k=2$. bipartite.

Let $m(k) \triangleq \min |E|$ in a k-uniform not 2-colorable hypergraph.

$m(2) = 3$.  $m(3) = 7$. (Fano plane).

$m(4) = 23$. unknown if $k \ge 5$.



Theorem (Erdős, 1964). $m(k) \geq 2^{k-1}$

Proof. consider $m < 2^{k-1}$ edges and a random 2-coloring.

$\Pr[\exists \text{ monochromatic edges}] \leq 2^{1-k} m < 1$. $\square$

Theorem (Erdős, 1964). $m(k) = O(k^2 2^k)$.

Proof. Fix $|V| = n$ TBD. Uniformly choose m edges from $\binom{[n]}{k}$

Given a coloring $\chi : V \to \{0, 1\}$. A_χ : χ is a proper coloring.

It suffices to show $\sum_\chi \Pr[A_\chi] < 1$. (why? probability space?)

If χ colors a vertices with 0, b vertices with 1. then for each edge e. $\Pr[e \text{ is monochromatic}] = \dfrac{\binom{a}{k} + \binom{b}{k}}{\binom{n}{k}} \geq 2 \dfrac{\binom{n/2}{k}}{\binom{n}{k}}$.

$\binom{n/2}{k} / \binom{n}{k} \geq \left(\dfrac{n/2 - k + 1}{n - k + 1}\right)^k = 2^{-k}\left(1 - \dfrac{k-1}{n-k+1}\right)^k \triangleq p$.

$\Pr[A_\chi] = (1 - \Pr[e \text{ is monochromatic}])^m \leq (1 - 2p)^m$

So $\sum_\chi \Pr[A_\chi] \leq 2^n (1 - 2p)^m < e^{n \ln 2 - 2mp}$. $n \ln 2 - 2mp < 0$ suffices.

Setting $n = k^2$. $p > 2^{-k}/e \Rightarrow m > n \ln 2 / (2p) = O(k^2 2^k)$. $\square$

List chromatic number, or choice number $ch(G)$.

list-coloring: a proper coloring where each vertex has a color list.

$ch(G)$: $\exists$ a proper coloring no matter what color lists are of size $ch(G)$.

$\chi(G) \le ch(G)$. but equality may not hold. $K_{3,3}$. colors: $\binom{[3]}{2}$

Lemma. If $\exists$ a non-2-colorable k-uniform hypergraph (n-edges), $ch(K_{n,n}) > k$.

Proof. Let $H = (V, E)$ k-uniform and $|E| = n$. Label vertices in $K_{n,n}$ by u_e and v_e, and assign color list e of size k.

If $K_{n,n}$ has a proper coloring. let C be the set of used colors among n vertices. Then color H by setting C 0 and $V \setminus C$ 1. $\square$

Corollary: $ch(K_{n,n}) > (1-o(1)) \log_2 n$ (since $m(k) = O(k^2 2^k)$).

Theorem. If $n < 2^{k-1}$. then $K_{n,n}$ is k-choosable ($ch(K_{n,n}) \le k$).

Proof. For each color. uniformly iid mark it L or R.

$\forall v \in$ left / right part of $K_{n,n}$ only choose L / R colors.

For each v. $\Pr[\text{no valid colors}] = 2^{-k}$. As long as $2n \cdot 2^{-k} < 1$.

$\Pr[\exists \text{ valid marking}] > 0$. A valid marking $\Rightarrow$ a proper coloring. $\square$

Corollary: $ch(K_{n,n}) = (1 \pm o(1)) \log_2 n$.

Theorem: $ch(G) > (1+o(1)) \log_2 d$ (by hypergraph containers). average degree.

Expectation. linearity of expectation.

Random variable: $X : \Omega \to \mathbb{R}$. event $A = X^{-1}(a)$.

Conditional expectation $\mathbb{E}[X \mid Y]$ $(f(y) = \mathbb{E}[X \mid Y = y])$.

Law of total expectation: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$

Averaging principle: $\mathbb{E}[X] = a \Rightarrow X \geq a$ / $X \leq a$ is possible.

Linearity of expectations: Let $X = c_1 X_1 + \cdots + c_n X_n$ Then:

$\mathbb{E}[X] = c_1 \mathbb{E}[X_1] + \cdots + c_n \mathbb{E}[X_n]$. Do Not Need Independence!

Question: what is the average # of fixed points of permutations?

A simple example: Hamiltonian paths in directed complete graphs
tournaments.

Theorem (Szele, 1943).

$\exists$ a tournament of size n with $\geq n!\,2^{-(n-1)}$ Hamiltonian paths.

Proof. Pick a random tournament. Let X = # of Hamiltonian paths

$\forall$ permutation π. $X_\pi = \begin{cases} 1 & \pi(1) \to \pi(2) \to \cdots \to \pi(n) \text{ is a path} \\ 0 & \end{cases}$

$X = \sum_\pi X_\pi \Rightarrow \mathbb{E}[X] = \sum \mathbb{E}[X_\pi] = n!\,2^{-(n-1)}$. $\square$

Remark. This was considered the first use of the probabilistic method. Szele conjectured that max # of Hamilton paths is $n!/(2-o(1))^n$, which was proved by Noga Alon in 1990.

Sum-free set (CMO 1995. National team training problem).

no a, b, c s.t $a+b=c$.

Theorem (Erdős 1965). $0 \notin S \subseteq \mathbb{N}$. $\exists$ sum-free subset of size $\geq \frac{|S|}{3}$.

Proof. For $\theta \in [0,1]$. $S_\theta = \{n \in S : \{n\theta\} \in (\frac{1}{3}, \frac{2}{3})\}$.

If $a+b=c$. then $a\theta + b\theta = c\theta$. But $(\frac{1}{3}, \frac{2}{3})$ is sum-free.

So S_θ is sum-free. Choose θ u.a.r. from $[0,1]$. $\{n\theta\}$ u.a.r.

$\Rightarrow \Pr[n \in S_\theta] = \frac{1}{3}$. By linearity, $\mathbb{E}[|S_\theta|] = |S|/3$. $\square$

Remark: Best know bound $\geq (n+2)/3$ by Jean Bourgain, 1997.

Crossing number $cr(G) \triangleq$ min # of crossings in a drawing of G.

Claim: If $G=(V,E)$ is planar. then $|E| \leq 3|V|-6$.

Proof. Recall the Euler's formula $v-e+f=2$. for every connected planar graph. compute # of incident pairs $(e,f) \Rightarrow 3f \leq 2e$.

$\Rightarrow |E| \leq 3|V|-6$ for all planar graphs. (may be disconnected.) $\square$

For each crossing, remove an edge incident to it. remaining a planar

$\Rightarrow (|E| - cr(G)) \leq 3|V|-6 \Rightarrow cr(G) \geq m-3n+6$. $\forall G$.

If $m = \Omega(n^2)$. then $cr(G) = \Omega(n^2)$? In fact $cr(G) = \Omega(n^4)$.

Pick a number $p \in (0,1)$. Select a induced subgraph $G' = (V', E')$.

$\forall v \in V$. $v \in V'$ with probability p. $\mathbb{E}[|V'|] = pn$, $\mathbb{E}[|E'|] = p^2 m$.

$\mathbb{E}[cr(G')] \le \mathbb{E}[\#\text{ of remaining crossings}] = p^4 cr(G)$.

$cr(G') \ge |E'| - 3|V'| + 6 \Rightarrow \mathbb{E}[cr(G') - (|E'| - 3|V'| + 6)] \ge 0$.

$\Rightarrow p^4 cr(G) - p^2 m + 3pn - 6 \ge 0 \Rightarrow cr(G) \ge p^{-3}(pm - 3n)$.

Setting $p = 4n/m$ (assume $m \ge 4n$) $\Rightarrow cr(G) \ge \frac{1}{64}\frac{m^3}{n^2}$.

Conjecture (Erdős – Guy, 1973) $cr(G) \ge c\, m^3/n^2$.

Theorem (Ajtai – Chvátal – Newborn – Szemerédi, 1982) $c = \frac{1}{100}$

Crossing lemma. crossing number inequality (Chazelle – Sharir – Welzl).

Applications of crossing lemma.

Incidence geometry. $I(P, L) = |\{(p, \ell) \in \underset{\text{points}}{P} \times \underset{\text{lines}}{L} : p \in \ell\}|$.

Example: $P = [k] \times [2k^2]$. $L = \{y = mx + b : m \in [k], b \in [k^2]\}$

Each line contains k points. Taking $n = k^3$. $I(P, L) = n^{4/3}$

Can we do better? Trivial bound $I(P, L) \le |P||L|$

Easy bound: every pair of points determine at most 1 line.

$$I(P, L)^2 = \Big(\sum_{\ell \in L} \sum_{p \in P} [p \in \ell]\Big)^2 \le |L| \cdot \sum_{\ell \in L} \Big(\sum_{p \in P} [p \in \ell]\Big)^2 \quad \text{Cauchy–Schwarz}$$

$$= |L| \cdot \sum_{p_1, p_2 \in P} \sum_{\ell \in L} [p_1 \in \ell] \cdot [p_2 \in \ell] \quad \text{if } p_1 \ne p_2 \ \exists \text{ unique } \ell.$$

$$\le |L| (I(P, L) + |P|^2) \quad \le |L| (|L| + 2|P|^2).$$

$\Rightarrow I(P,L) \lesssim |P||L|^{1/2} + |L| \quad I(P,L) \lesssim |L||P|^{1/2} + |P|. \quad n^{3/2}.$

Now applying crossing lemma. construct a graph $G = (V, E)$.

$V = P. \quad E = \{(P_1, P_2) : \exists \ell \in L,\ P_1, P_2 \in \ell$. and no points between them$\}$

$|E| = \sum_{\ell \in L} |P \cap \ell| - 1 \geq \frac{1}{2}(I(P,L) - |L|). \quad cr(G) \leq |L|^2$

So $|L|^2 \geq cr(G) \gtrsim \frac{|E|^3}{|V|^2} \gtrsim \frac{I(P,L)^3}{|P|^2}$ if $I(P,L) \geq 8|P|$.

Theorem (Szemerédi – Trotter, 1983) proof by Székely, 1997.

$I(P,L) \lesssim |P|^{2/3}|L|^{2/3} + |P| + |L| \quad (n^{4/3}$ if $|P| = |L| \approx n)$.