Lecture 5. Alteration. Second moment method.

Derandomization via expectation: finding independent sets.

Algorithmic proof of the Caro–Wei inequality: sorting vertices in degree-non-decreasing order. Assign each vertex weight $\frac{1}{\deg(v)+1}$

Greedily construct independent sets: at each step, take the first vertex and remove all its neighbours. then total weight removed ≤ 1.

Another algorithm based on conditional expectation: check vertices in an arbitrary order. For each vertex, there are 2 choices. Note that $\mathbb{E}[X] = \mathbb{E}[X \mid \mathcal{E}] \Pr[\mathcal{E}] + \mathbb{E}[X \mid \bar{\mathcal{E}}] \Pr[\bar{\mathcal{E}}]$ where X is the size of independent sets and $\mathcal{E}$ is the event of taking v. At least one of $\mathbb{E}[X \mid \mathcal{E}]$ and $\mathbb{E}[X \mid \bar{\mathcal{E}}]$ is not less than $\mathbb{E}[X]$.

Another example: Max Cut. For each vertex, uniformly mark 0 or 1.

Ramsey number revisit: Alteration.

Theorem (Erdős, 1947) If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, $R(k,k) > n$

Proof. Randomly color each edge. $\forall S \in \binom{[n]}{k}$, $\Pr[S \text{ 同色}] = 2^{1-\binom{k}{2}}$.

original: union bound   present: $\mathbb{E}[\# \text{ of 同色 } K_k] = \binom{n}{k} 2^{1-\binom{k}{2}}$.

Alteration: $\forall$ 同色 K_k delete a vertex from it. $\to$ no 同色 K_k

$\exists$ coloring s.t. # of 同色 $K_k \le \mathbb{E}[\#] \Rightarrow$ remaining size $\ge n - \mathbb{E}[\#]$.

Theorem. $\forall k, n$, we have $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$.

Remark. $R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k \cdot 2^{k/2}$ originally, now $\left(\frac{1}{e} + o(1)\right) k \cdot 2^{k/2}$

Analogously. $\forall s, t, n, p \in [0,1]$. $R(s,t) > n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}$.

Dominating set: $G = (V, E)$. $U \subseteq V$ dominating if $N^+(v) \cap U \neq \phi$.

Theorem. $\forall$ graph of size n with minimum degree $\delta > 1$ has a dominating set of size at most $\left(\frac{\log(\delta+1)+1}{\delta+1}\right) n$.

Naive attempt: take out vertices greedily. the first one remove $\delta+1$ vertices, but subsequent ones eliminate possibly fewer vertices.

Proof: alteration method: choose a random subset then add vertices.

Let $p \in [0,1]$ to be determined later. Independently pick each vertex with probability p into X, $Y = V \setminus N^+(X)$. then $X \cup Y$ dominating.

$\forall v \in V$, $\Pr[v \in Y] \le (1-p)^{1+\delta} \Rightarrow \mathbb{E}[|X \cup Y|] = \mathbb{E}[|X|] + \mathbb{E}[|Y|]$

$\le pn + (1-p)^{1+\delta} n \le (p + e^{-p(1+\delta)}) n \le \left(\frac{\log(\delta+1)+1}{\delta+1}\right) n$ by setting $p = \frac{\log(\delta+1)}{\delta+1}$ $\square$

Independent set: pick $v \in V$ with probability p. for each edge (u, v).

if both of them are chosen. remove anyone. $\mathbb{E} = pn - p^2 m \Rightarrow$

$\mathbb{E} \geq \frac{n^2}{4m}$ by setting $p = \frac{n}{2m}$. However, $\mathbb{E} \geq \frac{n^2}{2m+n}$ by Caro-Wei.

Graph with high girth (min length of cycles) and high chromatic number.

If a graph has a k-clique. the $\chi \geq k$. Conversely if χ is large is it always possible to verify it by observing local information?

Surprisingly, this is far from being true. even for "locally tree-like".

Theorem (Erdős, 1959) $\forall k, l$. $\exists$ graph with girth $> l$ and $\chi > k$.

Proof. Let $G \sim \mathcal{G}(n,p)$ with $p = (\log n)^2/n$. $\mathcal{G}(n,p)$: Erdős–Rényi.

Let X be # of cycles of length $\leq l$. In K_n, there are exactly $\binom{n}{i}(i-1)!/2$ cycles of length i. So $\mathbb{E}[X] = \sum_{i=3}^{l} \binom{n}{i} \frac{(i-1)!}{2} p^i$

$\leq \sum_{i=3}^{l} n^i p^i = o(n)$. By alteration, we now want to get rid of all short cycles. However we have another bad event: $\chi(G) \leq k$.

Proposition (Markov's inequality). $X \geq 0 \Rightarrow \forall a > 0$. $\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$.

Proof cont'd. By Markov's inequality. $\Pr[X \geq \frac{n}{2}] = o(1) < \frac{1}{2}$.

Note that $\chi(G) \geq \frac{n}{\alpha(G)}$. It suffices to show $\alpha(G) \leq n/k$. let

$t = \frac{3\log n}{p}$. $\Pr[\alpha(G) \geq t] \leq \binom{n}{t} (1-p)^{\binom{t}{2}} < n^t e^{-p(t-1)/2 \cdot t} = o(1) < \frac{1}{2}$.

Let n be sufficiently large s.t. $\Pr[X < \frac{n}{2}$ and $\alpha(G) < t] > 0$.

Remove a vertex from each cycle to get G'. Then $|V(G')| \geq \frac{n}{2}$, girth $> l$ and $\alpha(G') \leq \alpha(G) \leq \frac{3\log n}{p}$, thus $\chi(G') \geq \frac{np}{6\log n} > k$. $\square$

Remark: Construct such a graph is not easy. Example for Δ-free:

Let G_2 be graph K_2 (a single edge). Given $G_n = (V, E)$, construct $G_{n+1} = (V \cup V' \cup \{w\}, E \cup E')$ where V' is a copy of V. $\forall (u,v) \in E$ add (u', v) in E'. u' is the copy of u in V'. $\forall v' \in V'$, add (v', w) in E'. Then G_n is triangle-free and $\chi(G_n) = n$.

Local coloring: even observing εn vertices cannot certify high χ.

Theorem (Erdős 1962) $\forall k > 0$. $\exists \varepsilon > 0$ s.t. $\forall$ sufficiently large n $\exists G$ on n vertices with $\chi \geq k$ but $\chi(G[S]) \leq 3$ for all $|S| \leq \varepsilon n$.

Proof. Given k. let c, ε satisfy $c > 2k^2 H(1/k) \ln 2$, $\varepsilon < e^{-5} 3^3 c^{-3}$ where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the entropy function. Set $p = c/n$ and let $G \sim \mathcal{G}(n, p)$. Claim: G almost surely as desired.

① $\chi(G) > k$: If $\chi(G) \leq k$, then $\alpha(G) \geq n/k$. Let $X = \#$ of IS of size n/k. $\mathbb{E}[X] = \binom{n}{n/k}(1-p)^{\binom{n/k}{2}} < 2^{n(H(1/k)+o(1))} e^{-cn/2k^2(1+o(1))}$.

which is $o(1)$ by our condition on c $\Rightarrow \chi(G) > k$ almost surely.

② $\forall |S| \le \varepsilon n$. $\chi(G[S]) \le 3$: Otherwise. let S be a minimal set s.t. $\chi(G[S]) = 4$. Thus $\forall v \in S$. $\chi(G[S \setminus v]) = 3$. $\Rightarrow v$ has ≥ 3 neighbours in S. $\Rightarrow G[S \setminus v]$ has $\geq 3|S|/2$ edges. Let $t = |S|$.

$Pr[\exists$ size-t induced subgraph has $\geq 3t/2$ edges$] \le \binom{n}{t}\binom{\binom{t}{2}}{3t/2}\left(\frac{c}{n}\right)^{3t/2}$.

$\le \left(\frac{ne}{t}\left(\frac{te}{3}\right)^{3/2}\left(\frac{c}{n}\right)^{3/2}\right)^t \le \left(e^{5/2} 3^{-3/2} c^{3/2}\sqrt{t/n}\right)^t \triangleq P_t$. $\sum_{t \le \varepsilon n} P_t = o(1)$. $\square$

2-colorable hypergraph revisit. $m(k)$: min $|E|$ for non 2-colorable k-uniform hypergraph $H = (V, E)$. $2^{k-1} \le m(k) \le k^2 \cdot 2^k$

Theorem (Radhakrishnan & Srinivasan, 2000) $m(k) = \Omega\left(2^k \sqrt{\frac{k}{\log k}}\right)$.

Original proof contains a deterministic algorithm to switch colorings.

Proof (by Cherkashin & Kozik, 2015). Suppose $H = (V, E)$ with $|E| = m$. Map $V \to [0,1]$ and color vertices greedily from left to right. For each v, color it 0 unless forming an all-0 edge, o.w. color it 1. The resulting coloring has no all-0 edge. Bound $Pr[\exists$ all-1$]$.

Observation: $\forall$ color-1 v, $\exists e \in E$. s.t. v is the last one in e.

$\Rightarrow \forall$ all-1 edge f. $\exists e \in E$. s.t. the last of e = the first of f.

Call (e,f) conflicting. Let $[0,1] = L \cup M \cup R$ where $L = [0, \frac{1-p}{2})$.
$M = [\frac{1-p}{2}, \frac{1+p}{2}]$, $R = (\frac{1+p}{2}, 1]$. p to be determined. $\forall e \in E$,
$\Pr[\forall v \in e \text{ lies in } L/R] = (\frac{1-p}{2})^k \Rightarrow \Pr[\exists \text{ all } L \text{ or } R \text{ edge}] = 2m(\frac{1-p}{2})^k$
Suppose no all-L or all-R edges. If (e,f) conflicts, $e \cap f$ is a single vertex and lies in M. let $v = e \cap f$. $\Pr[v$ lies last in e, and lies first in $f] = \int_{(1-p)/2}^{(1+p)/2} x^{k-1}(1-x)^{k-1}\,dx \le p/4^{k-1}$. Thus
$\Pr[\exists \text{ conflicting } (e,f)] \le 2m(\frac{1-p}{2})^k + m^2 \frac{p}{4^{k-1}} < 2^{1-k} m e^{-pk} + (2^{1-k}m)^2 p$.
Setting $p = \log(2^{2-k}k/m)/k$. above $\Pr < 1$ for $m = \Omega(2^k \sqrt{\frac{k}{\log k}})$. $\square$

Recall Markov's inequality $\Pr[X \ge a] \le \frac{\mathbb{E}[X]}{a}$. Can we do better?

Theorem (Chebyshev's inequality). $\Pr[|X - \mathbb{E}[X]| \ge t] \le \frac{Var[X]}{t^2}$.
where variance $Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
Usually denoted by σ^2, and $\mathbb{E}[X]$ is usually denoted by μ.

Proof. $\sigma^2 = \mathbb{E}[(X-\mu)^2] \ge \Pr[|X-\mu| \ge t] \cdot t^2$ $\square$.

The use of Chebyshev's inequality is called the second moment method.

If $X = \sum X_i$. then $Var[X] = \sum_{i \ne j} Cov(X_i, X_j) + \sum_i Var[X_i]$

Distinct sums: $|S| = k$. all 2^k subset sums of S are distinct.

What is the minimum possible $\max S$. Example $S = \{1, 2, 4, \dots 2^{k-1}\}$.

Trivial bound: 2^k sums are distinct and $\leq k \max S$. $\Rightarrow \max S \geq \frac{2^k}{k}$.

Theorem: $\max S \gtrsim 2^k/\sqrt{k}$. $\left(\left(\sqrt{\frac{2}{\pi}} + o(1)\right) \frac{2^k}{\sqrt{k}}\right.$ by Dubroff – Fox – Xu.)

Proof. Let $S = \{x_1, \dots x_k\}$ and $n = \max S$. Choose $\varepsilon_i \in \{0, 1\}$ independently and u.a.r. Let $X = \sum \varepsilon_i x_i$ and $\mu = \mathbb{E}[X] = \frac{\sum x_i}{2}$.

The variance $\sigma^2 = \mathrm{Var}[X] = \frac{\sum x_i^2}{4} \leq \frac{nk^2}{4}$. By Chebyshev's inequality,

$\Pr[|X - \mu| < n\sqrt{k}] \geq \frac{3}{4}$. Since X takes distinct values for distinct $(\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k$, $\Pr[X = x] \leq 2^{-k}$ for all x. Thus we have

$\Pr[|X - \mu| < n\sqrt{k}] \leq 2^{-k} \cdot 2n\sqrt{k} \Rightarrow 2^{-k} 2n\sqrt{k} \leq \frac{3}{4}$. $\square$.

Application to analysis: Weierstrass approximation theorem.

Theorem (Weierstrass, 1885). Suppose $f: [0,1] \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$. Then $\exists$ polynomial $p(x)$ s.t. $\forall x \in [0,1]$ $|p(x) - f(x)| \leq \varepsilon$.

Proof (by Bernstein, 1912). Since $[0,1]$ is compact, f is uniformly continuously and bounded. W.l.o.g assume $|f(x)| \leq 1$. Also $\exists \delta > 0$ s.t. $|f(x) - f(y)| \leq \frac{\varepsilon}{2}$ for all $|x - y| \leq \delta$. Now we approximate f by $P_n(x) = \sum_{i=0}^{n} E_i(x) f(i/n)$, where n is sufficiently large, and

$E_i(x)$ peaks at $\frac{i}{n}$ and decays away from $\frac{i}{n}$. By setting $E_i(x) = \Pr[\mathrm{Bin}(n,x)=i] = \binom{n}{i} x^i (1-x)^i$, we have (since $\mathrm{Bin}(n,x)$ has mean nx and variance $nx(1-x) \le \frac{n}{4}$ then by Chebyshev's inequality) $\sum_{i:|i-nx|>n^{2/3}} E_i(x) = \Pr[|\mathrm{Bin}(n,x)-nx|>n^{2/3}] \le n^{-1/3}$. Note that $\sum_{i=0}^{n} E_i(x) = 1$. Taking $n > \max\{64\varepsilon^{-3}, \delta^{-3}\}$ we have $|P_n(x)-f(x)| \le \sum_{i=0}^{n} E_i(x) \left|f(\frac{i}{n})-f(x)\right| \le \sum_{|i-nx|\le n^{2/3}} E_i(x)\cdot\frac{\varepsilon}{2} + 2n^{-1/3} < \varepsilon$. $\square$