Lecture 7. $\chi(G(n,1/2))$. Lovász local lemma.

Consider $\omega(G(n,1/2)) = \alpha(G(n,1/2)) \geq n/\chi(G(n,1/2))$.

Let $f(k) = \mathbb{E}[X] = \binom{n}{k} 2^{-\binom{k}{2}}$ $f(k) \to \infty \Rightarrow \Delta^* \ll f(k) \Rightarrow \omega \geq k$.

Theorem. (Bollobás - Erdős, 1976 & Matula, 1976).

$\exists k \approx 2\log_2 n$ s.t. $\omega(G(n,1/2)) \in \{k, k+1\}$ w.h.p. $(1-o(1))$.

Proof. For $k = (1 \pm o(1)) 2\log_2 n$, $\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1} 2^{-k} = n^{-1+o(1)} = o(1)$.

Let $k_0 = k_0(n)$ be the value s.t. $f(k_0) \geq 1 > f(k_0+1)$. For n s.t.

$f(k_0) \to \infty$ and $f(k_0+1) \to 0$. it is known $\omega(G(n,1/2)) = k_0$ w.h.p.

If $f(k_0) = O(1)$. (or $f(k_0+1) = O(1)$. then let $k_0 = k_0+1$). we have

$f(k_0-1) \to \infty$ and $f(k_0+1) \to 0$. thus $\omega(G(n,1/2)) \in \{k_0-1, k_0\}$. $\square$

Remark. For constant p, $\alpha(G(n,p)) = \omega(G(n,1-p)) \approx 2\log_{1/p} n$ whp.

Clearly $\chi(G(n,1/2)) \geq n/\alpha(G(n,1/2)) \geq (1-o(1)) \frac{n}{2\log_2 n}$ w.h.p. $\leq$?

Theorem (Bollobás 1988). $\chi(G(n,1/2)) \approx \frac{n}{2\log_2 n}$ w.h.p $(1-o(1))$.

Proof idea: For the upper bound we give a strategy to properly color the graph. Take out an independent sets of size $\approx 2\log_2 n$. and color them with a new color. Until $o(n/\log_2 n)$ vertices remaining. color each of them a new color. However, after removing independent sets, the distributions of remaining subgraphs are no longer random graphs. Instead, if we fix a subset S of size m, the distribution of $G[S]$ induced by S is exactly $G(m,p)$. So we choose m and hope.

① w.h.p $\forall$ subset S of size m, $\alpha(G[S]) \gtrsim 2\log_2 m$.

② $n/(2\log_2 m) + m \le (1+o(1))\, n/(2\log_2 n)$.

For ②, $\log_2 m \geq (1-o(1))\log_2 n$ and $m = o(n/\log_2 n)$ suffices.

For ①, enumerate all size-m subsets and use the union bound.

On the one hand, concentration for ω is not enough, since $\binom{n}{m} o(1)$ is large. On the other hand, two points concentration is not necessary.

Theorem (Bollobás, 1988). Let k_0 be the largest number s.t. $f(k_0) \geq 1$.

Then $\Pr[\omega(G(n,1/2)) < k_0 - 3] = e^{-n^{2-o(1)}}$.

Proof of $\chi(G(n,1/2)) = (1+o(1))\frac{n}{2\log_2 n}$: Choose $m = n/(\log_2 n)^2$.

Fix any subset S of size m. $2\log_2 m = 2(\log_2 n - 2\log\log n)$.

$$\Pr[\alpha(G[S]) < (1-o(1))\, 2\log_2 n] \le e^{-m^{2-o(1)}} \ll e^{-n} \implies$$

$$\Pr[\forall S \text{ of size } m,\ \alpha(G[S]) \geq (1-o(1))\, 2\log_2 n] \geq 1 - \binom{n}{m} e^{-n} = 1 - o(1).$$

Set $k = k_0(m) - 3$. While $\geq m$ vertices remain, find an independent set of size k, color them and remove them. Finally color remaining vertices with distinct colors. $\Rightarrow \chi \leq \frac{n}{k} + m = (1+o(1))\frac{n}{2\log_2 n}$. $\square$

Chernoff bound and martingale concentration.

Chernoff bound: Let $S_n = X_1 + X_2 + \cdots + X_n$ where $X_i \in \{-1, 1\}$ uniformly iid. Then $\forall \lambda > 0$, $\Pr[S_n \geq \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$.

Proof. Let $t \geq 0$. consider the moment generating function $\mathbb{E}[e^{tS_n}]$

Setting $t = \lambda/\sqrt{n}$. $\Pr[S_n \geq \lambda\sqrt{n}] \leq \frac{\mathbb{E}[e^{tS_n}]}{e^{t\lambda\sqrt{n}}} \leq e^{-t\lambda\sqrt{n} + t^2 n/2} = e^{-\lambda^2/2}$ $\square$

Remark. Chebyshev only gives $\leq 1/\lambda^2$ since $Var[S_n] = \sum Var[X_i] = n$.

Corollary 1. Let $X_i \in [-1, 1]$ independently with $\mathbb{E}[X_i] = 0$ (not necessarily iid) Then $S_n = X_1 + \cdots + X_n$ has $\Pr[S_n \geq \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$.

Proof. By convexity, $e^{tx} \leq \frac{1-x}{2}e^{-t} + \frac{1+x}{2}e^{t}$. So $\mathbb{E}[e^{tX}] \leq \frac{e^{-t}+e^{t}}{2}$. $\square$

Corollary 2. Let X = sum of n independent Bernoulli's (not necessarily the same). Let $\mu = \mathbb{E}[X]$ and $\lambda \geq 0$. Then $\Pr[X \geq \mu + \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$.

Comparison to the normal distribution $N(0,1)$: $\mathbb{E}[e^{tX}] = e^{t^2/2}$. So

$\Pr[X \geq \lambda] \leq e^{-t\lambda}\mathbb{E}[e^{-tX}] = e^{-t\lambda + t^2/2} = e^{-\lambda^2/2}$ by setting $t = \lambda$.

A random variable X with $\Pr[|X| \geq t] \leq 2e^{-ct^2}$ for all $t \geq 0$ and constant $c > 0$ is called a sub-gaussian. such as sum of independent r.v.s. (usually the exact value of c is not significant).

We now develop similar sub-gaussian tail bound for other variables.

Definition (martingale). A martingale is a random variable sequence $\{Z_0, Z_1, \dots\}$. such that $\forall n$, $\mathbb{E}[Z_n] < \infty$, and $\mathbb{E}[Z_{n+1} \mid Z_0, \dots Z_n] = Z_n$.

Remark. Usually Z_n depends on $X_0 \dots X_n$, and $\mathbb{E}[Z_{n+1} \mid X_0, \dots, X_n] = Z_n$.

Doob martingale. Given underlying r.v.s $X_1 \dots X_n$, and $f(X_1, \dots X_n)$. then $Z_i = \mathbb{E}[f(X_1, \dots X_n) \mid X_1 \dots X_i]$ is a martingale w.r.t. $X_1 \dots X_n$.

Edge-exposure martingale: $\mathbb{E}[f(G(n,p)) \mid X_0, X_1, \dots, X_{\binom{n}{2}}]$ edges.

Vertex-exposure martingale: $\mathbb{E}[f(G(n,p)) \mid X_0, X_1, \dots, X_n]$ vertices.

Remark: There is a trade-off between the length and the difference bound.

Theorem (Azuma's inequality). Let $Z_0, Z_1, \dots Z_n$ be a martingale s.t. $|Z_i - Z_{i-1}| \leq c_i$, $\forall i \in [n]$. Then $\Pr[Z_n - Z_0 \geq \lambda] \leq e^{-\lambda^2 / 2(c_1^2 + \dots + c_n^2)}$.

More generally, if Z_i, conditioned on $Z_0, \dots Z_{i-1}$, lies inside an interval of length c_i. then $\Pr[Z_n - Z_0 \geq \lambda] \leq e^{-2\lambda^2/(c_1^2 + \dots + c_n^2)}$.

Remark. Applying Azuma to Z_n and $-Z_n$, it gives $\Pr[|Z_n - Z_0| \geq \lambda] \leq 2$.

Theorem (Bounded differences inequality). Let $X_1 \in \Omega_1, \ldots X_n \in \Omega_n$ be n independent r.v.s. Suppose $f: \Omega_1 \times \cdots \times \Omega_n \to \mathbb{R}$ is a function s.t.

$$|f(x_1, \ldots x_{i-1}, x_i, x_{i+1}, \ldots x_n) - f(x_1, \ldots x_{i-1}, y_i, x_{i+1} \ldots x_n)| \leq c_i$$

Then the random variable $Z = f(X_1, \ldots X_n)$ satisfies. $\forall \lambda \geq 0$.

$$\Pr[Z - \mathbb{E}[Z] \geq \lambda] \leq e^{-2\lambda^2/(c_1^2 + \cdots + c_n^2)} \quad \text{so is } \Pr[Z - \mathbb{E}[Z] \leq -\lambda]$$

In particular, if f is c-Lipschitz $|f(x) - f(y)| \leq c\|x - y\|_0$ then

$$\Pr[Z - \mathbb{E}[Z] \geq \lambda] \leq e^{-2\lambda^2/(nc^2)} \text{ and so is } \Pr[Z - \mathbb{E}[Z] \leq -\lambda].$$

Proof of $\omega(\mathcal{G}(n, 1/2))$: Let $k = k_0 - 3$. $Y = Y(G)$ be the maximum # of edge-disjoint set of k-cliques in G. $Y = f(X_{e_1}, \ldots X_{e_{\binom{n}{2}}})$.

Y changes ≤ 1 if G changes one edge (not true if G changes vertices).

By the bounded differences inequality, for $G \sim \mathcal{G}(n, 1/2)$. let $\mu = \mathbb{E}[Y]$.

$$\Pr[\omega(G) < k] = \Pr[Y(G) = 0] \leq \Pr[Y - \mu \leq -\mu] \leq e^{-2\mu^2/\binom{n}{2}}.$$

Our goal is $\Pr[\omega(G) < k] < e^{-n^{2-o(1)}}$. It suffices to show $\mu \geq n^{2-o(1)}$.

Consider an auxiliary graph H whose vertices are k-cliques in G. and $(u, v) \in E(H)$ if clique u and clique v overlap in ≥ 2 vertices in G.

Then $Y = \alpha(H) \geq \frac{|V(H)|^2}{|V(H)|+2|E(H)|}$ by Caro-Wei inequality.

Now use second moment method to compute $|V(H)|$ and $|E(H)|$.

$$\mu_v = \mathbb{E}[|V(H)|] = \binom{n}{k} 2^{-\binom{k}{2}} \geq n^{3-o(1)} \to \infty.$$

By the second moment method, $V(H) = (1 \pm o(1))\mu$ w.h.p.

$$\mu_e = \mathbb{E}[|E(H)|] = \frac{\Delta}{2} = \frac{\mu_v}{2}\Delta^* = \frac{\mu_v}{2}\sum_{l=2}^{k-1}\binom{k}{l}\binom{n-k}{k-l} 2^{\binom{l}{2}-\binom{k}{2}}$$

Let $g(l) = \binom{k}{l}\binom{n-k}{k-l} 2^{\binom{l}{2}-\binom{k}{2}}$. $\frac{g(l)}{g(l+1)} = \frac{(l+1)(n-2k+l+1)}{(k-l)^2} \cdot 2^{-l}$

Note that $k \approx 2\log_2 n$. $\Rightarrow$ If $l \geq \frac{3}{4}k$. then $g(l) \leq g(l+1)$.

$$\Rightarrow \sum_{\frac{3}{4}k \leq l < k} g(l) \leq \frac{k}{4} g(k-1) = \frac{k}{4} \cdot k \cdot (n-k) \cdot 2^{-(k-1)} = O(k^2/n).$$

If $l \leq \frac{3}{4}k$. $\frac{g(l)}{\mu_v} = \frac{\binom{k}{l}\binom{n-k}{k-l}}{\binom{n}{k}} \cdot 2^{\binom{l}{2}} \leq \frac{(n-k)^{k-l}/(k-l)!}{(n-k)^k/k!} \cdot \binom{k}{l} \cdot 2^{l(l-1)/2}$

$$\leq \frac{k^{2l}}{(n-k)^l} \cdot 2^{l(l-1)/2} = \left(\frac{k^2 \cdot 2^{(l-1)/2}}{n-k}\right)^l = \begin{cases} O(k^4/n^2) & l=2 \\ o(k^4/n^2) & l>2 \end{cases}$$

$\Rightarrow$ If $\mu_v \geq n^{3-o(1)}$. then $\mu_e = \frac{\mu_v}{2}\sum g(l) = c \cdot \mu_v^2 \cdot \frac{k^4}{n^2} \gg \mu_v$.

So $\mathbb{E}[Y] \geq \mathbb{E}\left[\frac{v^2}{v+2e}\right] \geq \mathbb{E}\left[\frac{v^2}{v+2e} \,\middle|\, v \geq (1-o(1))\mu_v\right] \Pr[v \geq (1-o(1))\mu_v]$

$= (1-o(1))\,\mathbb{E}\left[\frac{\mu_v^2}{\mu_v+2e}\right] \geq (1-o(1))\frac{\mu_v^2}{\mu_v+2\mu_e}$ (Jensen) $= O(n^2/k^4)$.

Without strong concentration. use alteration method. Pick each $v \in H$ with probability q. $\mathbb{E}[Y] \geq \mathbb{E}[q|V(H)| - q^2|E(H)|] = q\mu_v - q^2\mu_e$

Choose $q = \frac{\mu_v}{2\mu_e}$. $\Rightarrow \mathbb{E}[Y] \geq \mu_v^2/4\mu_e = O(n^2/k^4) = n^{2-o(1)}$. $\square$

Finally, if $\mathbb{E}[\chi(G(n,p))]$ is know. use vertex-exposure martingale.

Theorem (Shamir & Spencer 1987). $\forall \lambda \geq 0$. $\Pr[\chi - \mathbb{E} \substack{\geq \lambda\sqrt{n-1} \\ \leq -\lambda\sqrt{n-1}}] \leq e^{-2\lambda^2}$.

Remark. For $p = 1/2$. χ does not concentrate on any interval of length $\leq n^{1/4}$. But for sparse random graphs. $p = n^{-\alpha}$ for all $\alpha > 1/2$. χ has a two-point concentration. Leave a simple version as homework.

Summary. Suppose we have a set of events $A_1, \ldots, A_n$. each with probability p_i. If $\sum p_i < 1$. then by the union bound (or Markov's ineq). we know that $\Pr[\cap \bar{A}_i] > 0$ or even almost surely / w.h.p. / $1-o(1)$ if $\sum p_i = o(1)$. If $\sum p_i = O(1)$ or even $\sum p_i \to \infty$. then we know nothing about $\Pr[\cap \bar{A}_i]$. Let X_i be indicator of A_i. Hope $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$. Then $\Pr[\cap \bar{A}_i] = \Pr[X=0] = o(1)$. However if want > 0?

Lovász local lemma (Paul Erdős & Laszlo (Laci) Lovász, 1975).

Definition (Dependency). Suppose we have n "bad events" $A_1, \ldots A_n$. For each A_i. there is some subset $N(i) \subseteq [n]$ s.t. A_i is independent from $\{A_j : j \neq i, j \notin N(i)\}$. Here an event A_0 is independent from $\{A_1, \ldots A_m\}$ if $\forall B_i \in \{A_i, \bar{A}_i\}$. $\Pr[A_0 \mid B_1, B_2, \ldots, B_m] = \Pr[A_0]$.

Remark: we usually represent above relations by a dependency (di)graph. whose vertices are events, and $A_i \to A_j$ iff $j \in N(i)$. (or undirected).

Remark (Important). Independence $\neq$ Pairwise independence.

Consider $x_1, x_2, x_3 \in \{0,1\}$ uniformly and A_i is the event $\sum_{j \neq i} x_j = 0$.

Then any two events are pairwise independent but not independent.

Thus the empty graph is not a valid dependency graph.

But any graph with ≥ 2 edges is a valid dependency graph.

Example: (k-SAT. k-CNF). $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee x_5) \wedge \cdots$

a dependency graph. $\forall$ clauses $C_i \sim C_j$ if $\mathrm{vbl}(C_i) \cap \mathrm{vbl}(C_j) = \phi$.

Example: (hypergraph coloring). $H = (V, E)$. a coloring c is proper if $\forall e \in E$. $|c(e)| \geq 2$. dependency: $\forall e, f \in E$, $e \sim f$ if $e \cap f = \phi$.

Theorem (Lovász local lemma). Let $A_1, \ldots A_n$ be events with $\Pr[A_i] \leq p$. Suppose that each A_i is independent from a set of all other A_j except $\leq d$ of them. If $ep(d+1) \leq 1$. then $\Pr[\cap \bar{A}_i] > 0$.

Example (k-SAT). $\forall$ variable appear $\leq d$. satisfiable if $ekd2^{-k} \leq 1$.

Example. (coloring). $\forall$ vertex degree $\leq \Delta$. q-colorable if $ek\Delta q^{1-k} \leq 1$.