Lecture 12. Basic linear algebra method

Intersecting set family.

$\mathcal{F} \subseteq 2^{[n]}$ is an intersecting set family if $\forall A, B \in \mathcal{F}$, $A \cap B \neq \phi$.

Question: $\max |\mathcal{F}| = ?$ trivial 2^{n-1}

Erdős–Ko–Rado: $\mathcal{F}$ consists of k-element sets. $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Question: what if $|A \cap B| = k$? e.g. $k=1$. $\max |\mathcal{F}| = ?$

$|\mathcal{F}| = n$. $\mathcal{F} = \{\{1\}, \{1,2\}, \{1,3\}, \dots, \{1,n\}\}$. can it be greater?

Theorem (Fisher's inequality). Suppose $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting set family such that $|A \cap B| = k$ for every distinct $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \leq n$.

Proof. For each $S \in \mathcal{F}$, consider its incidence vector $v_S \in \mathbb{R}^n$, where the i-th element is 1 if $i \in S$ and 0 if $i \notin S$. Now we have:

$\langle v_S, v_S \rangle = |S|$ and $\langle v_S, v_T \rangle = k$ for any $S \neq T$.

Suppose $|\mathcal{F}| = m$. $M = \begin{pmatrix} v_{S_1}^T \\ \vdots \\ v_{S_m}^T \end{pmatrix} \in \mathbb{R}^{m \times n}$. Then $MM^T = \begin{pmatrix} |S_1| & & k \\ & \ddots & \\ k & & |S_m| \end{pmatrix}$

has full rank. Otherwise if $\exists c_1 \dots c_m$ not all zero, such that

$MM^T \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = 0$, we have $(|S_i| - k) c_i + k(c_1 + \dots + c_m) = 0$. $\forall i$.

That's impossible if all $|S_i| > k$, since $c_1 \dots c_m$ must have same sign.

If $|S_i| = k$ for some i. $S_i \subseteq S_j$ for all j and $S_j \setminus S_i$ are disjoint.

So $m-1 \leq n-k \leq n-1$. where we are done.

Now we have $\mathrm{rank}(MM^T) = m$. Thus $\mathrm{rank}(M) \geq m \Rightarrow m \leq n$. $\square$.

Question: can we construct $\mathcal{F}$ of size-n if $k > 1$?

Let q be a prime. $n = (q^r - 1)/(q-1)$. $k = (q^{r-2}-1)/(q-1)$. $V = \mathbb{F}_q^r$ be a linear space of dimension r. Then V has exactly n one dimensional subspaces ($\forall v \neq 0 \in V$. $|\mathrm{span}\{v\}| = q$ and $0 \in \mathrm{span}\{v\}$) and n $(r-1)$-dimensional subspaces. Now let $S_i \subseteq [n]$ such that $j \in S_i$ if the j-th one dimensional subspace is a subspace of the i-th $(r-1)$ dimensional. Thus $|S_i| = (q^{r-1}-1)/(q-1)$ and $|S_i \cap S_j| = k$.

($\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W)$ if U. W are subspaces).

Odd/even town.

n citizens in odd/even town are building clubs as follows:

(1) each club contains an odd number of members

(2). each pair of clubs shares an even number of members

Goal: making the number of clubs as large as possible.

Oddtown theorem: There are at most n clubs.

Proof. Consider incidence vectors and incidence matrix M. Let $Q = MM^T$

Assume there are m clubs. $Q \in \mathbb{R}^{m \times m}$ has full rank. why?

Exactly one term of $\det(Q)$ is odd. or consider Q over $\mathbb{F}_2$.

$Q = I$ so Q has full rank over $\mathbb{F}_2$. Thus $n \geq \operatorname{rank}(M) \geq m$. $\square$

Example. $S_i = \{i\}$. $|S_i \cap S_j| = 0$.

If n is even. then let $S_i = [n] \setminus \{i\}$. $|S_i \cap S_j| = n-2$.

Exercise: even/odd town problem.

Even/even town.

Eventown theorem: Let $\mathcal{F}$ be a set family of even size subsets of $[n]$. If $\forall A. B \in \mathcal{F}$. $|A \cap B|$ is even. then $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$

Example. n is even. $n/2$ pairs $\{1,2\}, \{3,4\} \ldots \{n-1, n\}$.

for each pair. either takes both of them, or take neither.

Proof. Consider incidence vectors over $\mathbb{F}_2$. $\forall A. B \in \mathcal{F}$. $\langle v_A, v_B \rangle = 0$.

Let $W = \operatorname{span}\{v_S : S \in \mathcal{F}\}$ be a subspace of $\mathbb{F}_2^n$. $\forall v_S. v_T \in W$

$\langle v_S, v_T \rangle = 0$. So $W \subseteq W^\perp$. $\dim(W) + \dim(W^\perp) = n \Rightarrow |W| \leq 2^{\lfloor n/2 \rfloor}$ $\square$

In general. we have the following result:

Theorem (Ahlswede - El Gamal - Pang, 1984).

Let $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2$ be two set families over $[n]$. $\forall S_1 \in \tilde{\mathcal{F}}_1$, $S_2 \in \tilde{\mathcal{F}}_2$, $|S_1 \cap S_2|$ is even. Then $|\tilde{\mathcal{F}}_1| \cdot |\tilde{\mathcal{F}}_2| = 2^n$.

Exercise: odd/odd town problem

Graph decomposition: Partition K_n into disjoint complete bipartite graphs.

Upper bound: $n-1$ parts is possible. $K_n \setminus K_{1,n-1} = K_{n-1}$.

Theorem (Graham - Pollark, 1972). If K_n is decomposed into m edge-disjoint complete bipartite graphs. then $m \geq n-1$.

Proof (due to Trevberg, 1982) Let the vertex set of K_n be $[n]$, and vertex sets of m complete bipartite graphs are $(L_1, R_1) \cdots (L_m, R_m)$.

Suppose $m < n-1$, Consider the system of linear equations:

$x_1 + x_2 + \cdots + x_n = 0$. $\sum_{k \in L_i} x_k = 0$. $i = 1 \ldots m$.

There are n variables and $m+1 < n$ equations. So $\exists$ nontrivial solution $c_1 \cdots c_n$. However, $\sum_{i<j} c_i c_j = \sum_{k=1}^{m} \left(\sum_{i \in L_k} c_i \cdot \sum_{j \in R_k} c_j \right) = 0$.

Thus $\sum c_i^2 = (\sum c_i)^2 - 2 \sum_{i<j} c_i c_j = 0$. contradicts nontrivial. $\square$

Exercise: How about decomposition of complete subgraphs?

Space of polynomials.

Let S be a set of points in $\mathbb{R}^n$. $D(S) = \{\|u-v\| : u, v \in S\}$.

Hope $D(S)$ as small as possible. If $|D(S)| = 1$. $|S| = n+1$ is possible.

If $|D(S)| = 2$? Let $S = \{v \in \{0,1\}^n : \|v\|_1 = 2\}$. $|S| = \binom{n}{2}$.

Theorem (Larman – Rogers – Seidel, 1977)

Every two-distance set in $\mathbb{R}^n$ has size at most $\frac{1}{2}(n+1)(n+4)$

Proof. Suppose $D(S) = \{\delta_1, \delta_2\}$. For each $1 \le i \le |S|$ define a polynomial

$$f_i(x) : \mathbb{R}^n \to \mathbb{R} = (\|x - v_i\|^2 - \delta_1^2)(\|x - v_i\|^2 - \delta_2^2)$$

Each polynomial has n variables and degree 4. So there are at most

n^4 polynomials if they are linearly independent.

Claim: Let $f_1 \dots f_m$ be functions $\Omega \to \mathbb{F}$. and $x_1 \dots x_m \in \Omega$.

If $f_i(x_j) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i > j \end{cases}$. then $f_1 \dots f_m$ are linearly independent

in the space $\mathbb{F}^\Omega$. (upper triangular criterion)

Proof of criterion: Suppose there are nontrivial $\lambda_1 \dots \lambda_m$ such that

$$f = \lambda_1 f_1 + \cdots + \lambda_m f_m \equiv 0.$$

Let k be the smallest i such that $\lambda_i \neq 0$. Then

$$f(x_k) = \lambda_k f_k(x_k) + \lambda_{k+1} f_{k+1}(x_k) + \cdots + \lambda_m f_m(x_k) \neq 0. \quad \square$$

Back to two-distance problem. $f_i(v_i) \neq 0$, but $f_i(v_j) = 0$ if $i \neq j$.

So $f_1 \dots f_m$ are linearly independent. Thus $m \leq n^4$???

Expanding $f_i(x)$, we find f_i is a linear combination of

$\left(\sum_{k=1}^{n} x_k^2\right)^2$, $x_j\left(\sum_{k=1}^{n} x_k^2\right)$, $x_j x_k$, x_j and 1.

Thus $m \leq 1 + n + \binom{n}{2} + n + 1 = \frac{1}{2}(n+1)(n+4)$. $\square$

Improvement (Blokhuis, 1981) $m \leq \binom{n+2}{2}$.

Proof: $f_1 \dots f_m, x_1, \dots, x_n, 1$ are linearly independent.

So $m \leq \frac{1}{2}(n+1)(n+4) - n - 1 = \binom{n+2}{2}$. $\square$

Generalization of Fisher inequality: $\forall A, B \in \mathcal{F}$, $|A \cap B| \in L$

Theorem (Frankl–Wilson, 1981)

If $\mathcal{F}$ is an L-intersecting set family, then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

Remark. Let $L = \{0, 1, \dots, l-1\}$. Note that the family of all subsets of size at most l is L-intersecting. So the bound is optimal.

Proof. Let $\mathcal{F} = \{S_1, \dots, S_m\}$, $|S_1| \leq \cdots \leq |S_m|$ and $L = \{l_1, \dots, l_t\}$.

For each S_i associate an incidence vector v_i of length n.

$v_{i,j} = \begin{cases} 1 & \text{if } j \in S_i \\ 0 & \text{if } j \notin S_j \end{cases}$. Clearly $\langle v_i, v_j \rangle = |S_i \cap S_j| \in L$.

Define polynomials $f_i(x): \mathbb{R}^n \to \mathbb{R}$. $= \prod_{k: l_k < |S_i|} (\langle v_i, x \rangle - l_k)$.

Note that $f_i(v_i) = \prod_{k: l_k < |S_i|} (|S_i| - l_k) > 0$ and $\forall j \neq i$.

$f_i(v_j) = \prod_{k: l_k < |S_i|} (\langle v_i, v_j \rangle - l_k) = 0$. By triangular criterion.

$f_1, \dots, f_m$ are linearly independent.

Moreover, we can only consider $\{0, 1\}$ vectors. so $x_i^k = x_i$. Replace

all higher power x_i^k by x_i, which does not change linear independence.

Now polynomials are generated by $\prod_{i \in I} x_i$, where $I \subseteq [n]$ and $|I| \le t$.

The number of such monomials is exactly $\sum_{k=0}^{t} \binom{n}{k}$. $\square$

Theorem (Deza–Frankl–Singhi, 1983).

Let L be a set of integers and p be a prime number. Assume $\mathcal{F} = \{S_1, \dots, S_m\}$ is a set family such that

① $|S_i| \notin L \bmod p$, for all $1 \le i \le m$;

② $|S_i \cap S_j| \in L \bmod p$, for all $1 \le i < j \le m$.

Then $|\mathcal{F}| \le \sum_{k=0}^{|L|} \binom{n}{k}$.

Constructive Ramsey graphs.

Ramsey graph: G has neither clique nor independent set of size t.

Lemma. Let $G=(V,E)$ be a graph, where $V=\binom{[k]}{3}$, and $(A,B)\in E$ if $|A\cap B|=1$. Then G has neither clique or IS of size $> k$.

Proof. Let Q be a clique. $\forall A, B\in Q$, $|A\cap B|=1$, so $|Q|\le k$.

Let S be an independent set. $\forall A, B\in S$, $|A\cap B|=0$ or 2.

So $|S|\le k$ by the odd-town theorem. $\square$

Theorem (Frankl 1977, Frankl–Wilson 1981).

Let p be a prime number, $k=p^3$, $G=(V,E)$ where $V=\binom{[k]}{p^2-1}$ and $(A,B)\in E$ if $|A\cap B|\not\equiv -1 \bmod p$. Then G has neither a clique or an IS of size more than $t=\sum_{i=0}^{p-1}\binom{k}{i}$.

Proof. Let Q be a clique. Then Q is an $\{0,1,\dots,p-2\}$-intersecting set family (mod p), $\forall U\in Q$, $|U|=p^2-1\equiv -1 \bmod p$. So $|Q|\le t$.

Let S be an IS. $\forall A, B\in S$, $|A\cap B|\equiv -1 \bmod p$. So $|A\cap B|\in \{p-1, 2p-1, \dots, p^2-1\}$. Thus $|S|\le t$. $\square$

Corollary. $n=t^{\Omega(\ln t/\ln\ln t)}$ for $p=\Omega(\ln t/\ln\ln t)$.