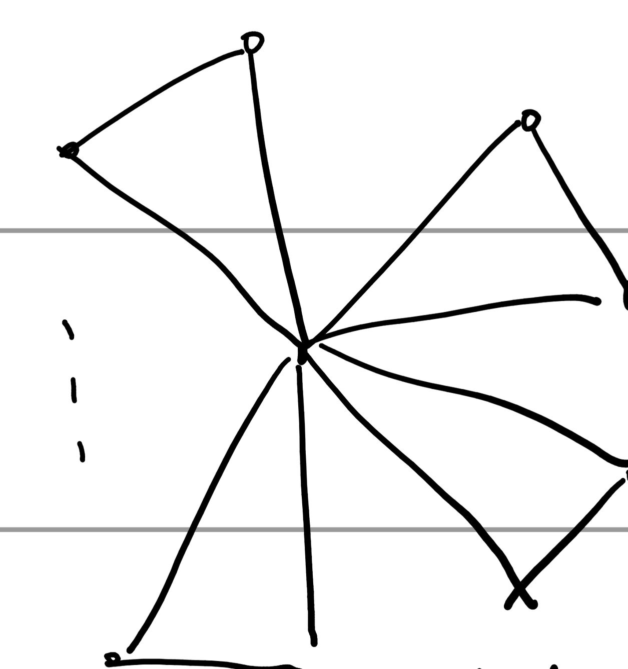
Lecture 13. Eigenvalues of graphs.

Friendship theorem. (Erdős - Rényi - Sós. 1966).

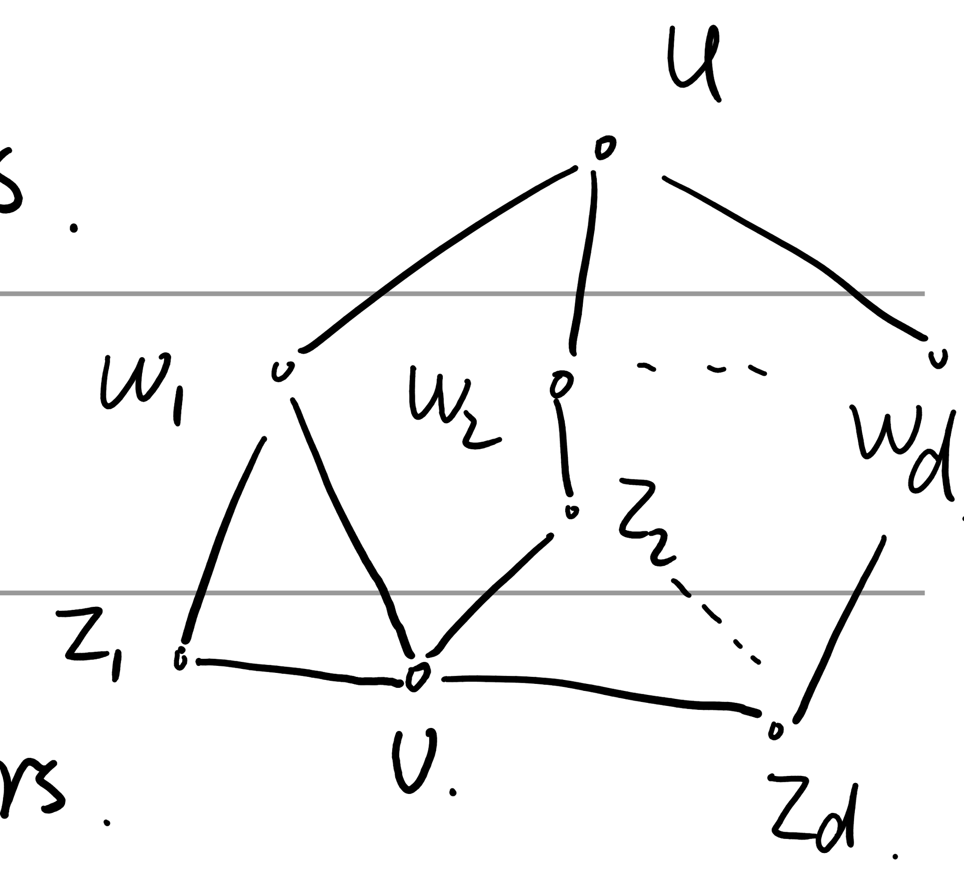
Suppose that G is a finite graph in which any two vertices have exactly one common neighbor. Then G is a windmill graph.

Remark. Note that this theorem is not true for infinite graphs. We may start from a 5-cycle. and repeatedly add common neighbors for all pairs of vertices that don't have one. This leads to an infinite counterexample.


windmill.

Proof. Assume it is false and G is a counterexample.

So no vertex in G is adjacent to all other vertices.



Step 1. $\forall u \not\sim v$. $\deg(u) = \deg(v)$.

Suppose $\deg(u) = d$. and $w_1 \dots w_d$ are its neighbors.

u, v have a common neighbor. w.l.o.g. assume it is w_1.

$\forall w_k$. w_k, v have a common neighbor z_k. how many distinct z_k?

$\forall i \neq j$. $z_i \neq z_j$. otherwise (u, z_i) have two common neighbors w_i and w_j. also $u \neq z_i$ since $u \not\sim v$. so all z_k are distinct.

But $w_1 = z_k$ for some k. so $|\{w_1, z_1, \dots z_d\}| = d$.

$\deg(v) \geq d = \deg(u)$. By symmetry. $\deg(v) \leq \deg(u)$.

Step 2. G is regular.

Assume $u \sim w \sim v$. and $u \nsim v$. any vertex different from w is not adjacent to u or v. so has degree d. But w also has a nonneighbor. So $\deg(w) = d$. G is d-regular.

Step 3. G has $d^2 - d + 1$ vertices.

u has d neighbors $w_1 \cdots w_d$. each of them has $d-1$ neighbors other than u. none of them are the same vertex. otherwise it shares two neighbors with u. Moreover, each w_i has exactly one neighbor w_j in u's neighbors. So there are $d(d-2) + d + 1$ vertices.

All vertices in G have been counted. So G has $n = d^2 - d + 1$ vertices.

Step 4. contradicts if $d \geq 3$. ($d = 2$ is trivial since $G = \Delta$)

Let A be the adjacency matrix of G. Then $A^2 = \begin{pmatrix} d & & 1 \\ & d & \\ 1 & & \ddots \ d \end{pmatrix}$

$= J + (d-1) I$. J has a trivial eigenvalue n.

Moreover. $\operatorname{rank}(J) = 1$. $\dim \ker J = n-1$ So J has eigenvalue 0 of multiplicity $n-1$. Thus A^2 has eigenvalues $(d^2, d-1, \dots, d-1)$.

A is a real symmetric matrix. So A has n eigenvalues

$\lambda_1 \dots \lambda_n$. where $\lambda_1^2 = d^2$, $\lambda_2^2 = \dots = \lambda_n^2 = d-1$.

Clearly $\lambda_1 = d$ since G is d-regular.

Suppose there are s $\sqrt{d-1}$ and r $-\sqrt{d-1}$ in $\lambda_2 \dots \lambda_n$.

Then $s+r = n-1$, and $(s-r)\sqrt{d-1} + d = 0$ $(\lambda_1 + \dots + \lambda_n = Tr(A))$.

Thus $\sqrt{d-1} = \frac{d}{r-s}$. Assume $d-1 = c^2$. So $c = \frac{c^2+1}{r-s}$

But $\gcd(c, c^2+1) = 1$. Hence $c = 1$ and $d = 2$. contradiction. $\square$

Remark. Another characterization: between any two vertices there is exactly one path of length 2.

Kotzig's conjecture: Let $l \geq 2$. There is no finite graph with the property that between any two vertices there is exactly one path of length l. (proved up to $l \leq 33$. originally $l \leq 8$ by Kotzig).

Some facts of eigenvalues (spectral graph theory).

Eigenvalues are roots of $\det(A - \lambda I) = 0$. So there are n eigenvalues.

Some of them may be complex.

If A is real symmetric, then all eigenvalues are real.

$\sum \lambda_i = \mathrm{Tr}(A) = \sum a_{ii}$. $\prod \lambda_i = \det(A)$.

# of nonzero eigenvalues including multiplicity is $\mathrm{rank}(A)$.

Proposition. $\sum \lambda_i = 0$ if the graph has no loops.

Proposition. If G is d-regular, then d is the largest eigenvalue.

Moreover, if G has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then

$\bar{G}$ has eigenvalues $(n-1-d, -1-\lambda_2, -1-\lambda_3, \cdots -1-\lambda_n)$.

The eigenvector corresponding to d is $\mathbb{1} = (1, 1, \cdots, 1)$.

So if $A \cdot v = \lambda_i v$ for some $i \neq 1$.

$A(\bar{G})v = (J - I - A(G))v = (0 - 1 - \lambda_i)v$.

Proposition. K_n has adjacency matrix $A = J - I$.

The eigenvalues are $n-1$ and -1 of multiplicity $n-1$.

Proposition. $K_{m,n}$ has adjacency matrix A of rank 2.

The eigenvalues are $\pm\lambda$ and 0 of multiplicity $n-2$.

If $Av = \lambda v$, v has m coordinates equal to α and n coordinates

equal to β. So $Av = (m\beta, \cdots m\beta, n\alpha, \cdots n\alpha)$.

Thus $m\beta = \lambda\alpha$ and $n\alpha = \lambda\beta$. $\lambda = \sqrt{mn}$.

Proposition. Suppose $A \in \mathbb{R}^{n\times n}$ real symmetric. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Then $\lambda_1 = \max\limits_{v\neq 0} \frac{v^TAv}{v^Tv}$. $\lambda_2 = \max\limits_{v\perp u} \frac{v^TAv}{v^Tv}$. where $Au = \lambda_1 u$.

In general. $\lambda_k = \max\limits_{\dim V=k} \min\limits_{v\in V} \frac{v^TAv}{v^Tv} = \min\limits_{\dim U=k-1} \max\limits_{v\perp U} \frac{v^TAv}{v^Tv}$ $(v\neq 0)$

Proof. Assume $v^Tv = 1$. We only prove the first inequality.

A is real symmetric. So there exists an orthonormal basis of eigenvectors $u^1, u^2, \ldots, u^n$ s.t. $Au^i = \lambda_i u^i$. $\begin{cases} \langle u^i, u^j\rangle = 0 \\ \langle u^i, u^i\rangle = 1 \end{cases}$

Suppose $v = \sum_{i=1}^n \alpha_i u^i$. Then $v^Tv = \sum_{i,j} \alpha_i\alpha_j \langle u^i, u^j\rangle = \sum \alpha_i^2$.

$v^TAv = (\sum \alpha_i u^i)^T A (\sum \alpha_i u^i) = \sum_{i,j} \alpha_i\alpha_j\lambda_j \langle u^i, u^j\rangle = \sum \lambda_i\alpha_i^2$

Now consider a subspace V generated by the first k eigenvectors:

$V = \text{span}\{u^1, \ldots, u^k\}$. For any $v\in V$. $v^TAv = \sum_{i=1}^k \lambda_i\alpha_i^2 \geq \lambda_k \sum \alpha_i^2$.

Thus $\max\limits_{\dim V=k} \min\limits_{v\in V} \frac{v^TAv}{v^Tv} \geq \lambda_k$.

On the other hand, consider any subspace V of dimension k. Let W be the subspace generated by the last $n-k+1$ eigenvectors:

$W = \text{span}\{u^k, \ldots, u^n\}$. $\dim W = n-k+1$. So $V\cap W \neq \phi$.

Let $w \in V\cap W = \sum_{j=k}^n \beta_j u^j$. Then $w^TAw = \sum_{j=k}^n \lambda_j\beta_j^2 \leq \lambda_k \sum \beta_j^2$

Thus $\max\limits_{\dim V=k} \min\limits_{v\in V} \frac{v^TAv}{v^Tv} \le \lambda_k$. $\square$

Theorem (Hoffman bound).

Suppose G is a d-regular graph with eigenvalues $d=\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Then $\alpha(G) \le \frac{n}{1-d/\lambda_n}$.

Remark. Note that $\lambda_n < 0$. So the denominator is greater than 1.

Proof. Let $G=(V,E)$ and $S\subseteq V$ be a maximum independent set.

Let $v = n\mathbb{1}_S - \alpha\mathbb{1}$ be the indicator vector of S, modified to be orthogonal to $\mathbb{1}$. Clearly $\lambda_n \le \frac{v^TAv}{v^Tv}$.

$$v^Tv = (n\mathbb{1}_S - \alpha\mathbb{1})^T(n\mathbb{1}_S-\alpha\mathbb{1}) = n^2\alpha + \alpha^2 n - 2n\alpha^2 = \alpha n(n-\alpha).$$

$$v^TAv = n^2\mathbb{1}_S A\mathbb{1}_S + \alpha^2\mathbb{1}A\mathbb{1} - 2n\alpha\mathbb{1}_S A\mathbb{1}.$$

$$= n^2\sum_{i,j}[i\in S]A_{ij}[j\in S] + \alpha^2\cdot dn - 2n\alpha d\alpha$$

$$= -\alpha^2 dn.$$

Thus $\lambda_n \le \frac{-\alpha d}{n-\alpha}$, which is $\alpha \le \frac{n}{1-d/\lambda_n}$ $\square$.

Example. K_n: $\lambda_n=-1$. $d=n-1$. $\alpha\le 1$.

$K_{n,n}$: $\lambda_{2n}=-n$, $d=n$. $\alpha\le n$.

Petersen graph: $\lambda_{10}=-2$, $d=3$. $\alpha\le 4$.

Petersen graph and Moore graph.

The Peterson graph:

- the smallest bridgeless 3-regular graph with no 3-edge coloring.
- the smallest 3-regular graph of girth 5.
- the largest 3-regular graph of diameter 2.
- having 2000 spanning trees. the most of 3-regular 10-vertices graph.

Eigenvalues: $(3, 1, 1, 1, 1, 1, -2, -2, -2, -2)$.

strongly regular. each vertex has same degree (3).

$\forall (u,v) \in E$ has same number of shared neighbors (0)

$\forall (u,v) \notin E$ has same number of share neighbors. (1).

So $(A^2)_{ii} = 3$, $(A^2)_{ij} = 1 - A_{ij}$ if $i \neq j$. $\Rightarrow A^2 + A - 2I = J$.

Now consider eigenvalues λ other than $d=3$. $Av = \lambda v$. $v \perp \mathbb{1}$.

So $(A^2 + A - 2I)v = \lambda^2 v + \lambda v - 2v = Jv = 0$. Thus $\lambda^2 + \lambda - 2 = 0$.

Theorem. There is no decomposition of K_{10} into 3 copies of Petersen.

Proof. Suppose exists. Let A. B. C be adjacency matrices of 3 copies.

Then $A + B + C = J - I$. Let V_A. V_B be the subspaces spanned by eigenvectors corresponding to eigenvalue 1 for A and B.

$\dim(V_A) = \dim(V_B) = 5$. and $V_A \perp \mathbb{1}$. $V_B \perp \mathbb{1}$. So $V_A \cap V_B \neq \phi$.

Therefore $\exists\, w \neq 0 \in V_A \cap V_B$. Clearly $w \perp \mathbb{1}$. i.e. $Jw = 0$.

Thus. $Cw = (J - I - A - B) w = -w - Aw - Bw = -3w$.

But -3 is not an eigenvalue of C. contradiction. $\square$.

Moore graph: d-regular graph of diameter k and girth $2k+1$.

(the Peterson graph is an unique Moore graph of $d=3$, $k=2$).

Theorem. (Hoffman – Singleton. 1960).

If G is d-regular of diameter 2, girth 5. then $d \in \{2, 3, 7, 57\}$.

Proof. Let $G = (V, E)$. $n = |V| = 1 + d + d(d-1) = d^2 + 1$.

$\forall (u,v) \in E$, u, v do not share a common neighbor.

$\forall (u,v) \notin E$, $u.v$ share precisely one common neighbor.

So $A^2 + A - (d-1) I = J$. Any eigenvalue $\lambda \neq d$ has $\lambda^2 + \lambda = d-1$.

Let $r = \sqrt{4d-3}$. Then $\lambda = \frac{1}{2}(-1 \pm r)$. Assume multiplicity a. b.

We have $d - \frac{1}{2}(a+b) + \frac{r}{2}(a-b) = 0$. Also. $a+b = n-1 = d^2$.

Thus $(a-b) r = d^2 - 2d$. $\Rightarrow \begin{cases} a = b \\ d = 2 \end{cases}$ or $4d - 3 = c^2$.

Since $d = (c^2+3)/4$. $c^5 + c^4 + 6c^3 - 2c^2 + (9-32a)c = 15$.

c is an integer. hence $c \in \{1, 3, 5, 15\}$. giving $d \in \{1, 3, 7, 57\}$. $\square$

Remark: It is still open whether $d = 57$ is possible.