2 Random Variables, Expectation, Running Times of Algorithms

2.1 Random variables and expectations

Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space, where Ω is called the sample space, or the set of outcomes, and $\mathcal{F}$ is a σ-algebra representing the set of "valid" events.

A random variable is a function that assigns a numerical value to each possible outcome of a random experiment.

A random variable is neither random nor a variable.

Definition 2.1. Random variable

A **random variable** X is a ($\mathcal{F}$-measurable) function that maps outcomes from a sample space Ω to real numbers:

$$X : \Omega \to \mathbb{R}$$

A function f is $\mathcal{F}$-measurable if for any x, $f^{-1}(x) \in \mathcal{F}$.

Example 2.1.

Let X be the result of rolling a fair 6-sided die. Then:

$$X \in \{1,2,3,4,5,6\}, \quad \mathbb{P}(X = i) = \frac{1}{6} \text{ for } i = 1, \dots, 6$$

Definition 2.2. Expectation

The **expected value** (or expectation) of a discrete random variable X is:

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$$

For continuous random variables, replace the sum with an integral, namely, for a continuous random variable X with probability density function $f_X(x)$, the expectation is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \, \mathrm{d}x$$

Example 2.2.
For the die roll random variable:

$$\mathbb{E}[X] = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6} = 3.5$$

A key property of expectations is the *linearity of expectation*.

Theorem 2.1. Linearity of Expectation
For any random variables X, Y and constants $a, b \in \mathbb{R}$:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

This holds *even if X and Y are dependent*.

Proof. For discrete random variables:

$$\begin{aligned}
\mathbb{E}[aX+bY] &= \sum_x \sum_y (ax+by)\mathbb{P}(X=x, Y=y) \\
&= a\sum_x x \sum_y \mathbb{P}(X=x, Y=y) + b\sum_y y \sum_x \mathbb{P}(X=x, Y=y) \\
&= a\sum_x x\mathbb{P}(X=x) + b\sum_y y\mathbb{P}(Y=y) \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y]
\end{aligned}$$

In general, let $Z = aX + bY$. The expectation is:

$$\begin{aligned}
\mathbb{E}[Z] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= a\int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\right)\mathrm{d}x + b\int_{-\infty}^{\infty} y\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\right)\mathrm{d}y \\
&= a\int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y]
\end{aligned}$$

□

Example 2.3.
(Independent) Let X and Y be two independent die rolls. The expected value of their sum:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3.5 + 3.5 = 7$$

(Dependent) Let X be a die roll and $Y = X$. Despite dependence:

$$\mathbb{E}[X+Y] = \mathbb{E}[2X] = 2\mathbb{E}[X] = 7$$

Remark 2.1.
Linearity of expectations can be generalized to any finite terms. For any finite collection of random variables $X_1, \dots, X_n$ and constants $a_1, \dots, a_n \in \mathbb{R}$:

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

This holds regardless of dependence between the X_i. But when the number of terms is random or infinite, linearity might not hold.
For random number of terms, suppose $X_1 = X_2 = \cdots = X_N = N$ where N is random in $\{1, \dots, 6\}$. Then

$$\mathbb{E}[N] \cdot \mathbb{E}[X_1] = 12.25 \neq \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[N \cdot N] = 15.166...$$

We will introduce Wald's equation to address this case in the future.
For infinite number of terms, consider the *St. Petersburg paradox*. In each stage of the game, a fair coin is tossed and a gambler guesses the result. He wins the amount he bet if his guess is correct and loses the money if he is wrong. He bets 1 dollar at the first stage. If he loses, he doubles the money and bets again. The game ends when the gambler wins. In stage i, he wins X_i with $\mathbb{E}[X_i] = 0$, so $X = \sum_{i=1}^{\infty} \mathbb{E}X_i = 0$. On the otherhand, he eventually wins 1 dollar, so $\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = 1 \neq \sum_{i=1}^{\infty} \mathbb{E}X_i$. In fact, when infinite sums are involved, additional conditions are required. One of such condition is absolute convergence, namely

$$\mathbb{E}\sum_{i=1}^{\infty} |X_i| < \infty \quad \text{or} \quad \sum_{i=1}^{\infty} \mathbb{E}|X_i| < \infty.$$

2.2 Coupon Collector

Consider the following coupon collector problem. Given n coupons, how many coupons one expects to draw with replacement before having drawn each coupon at least once? Formally, given n distinct coupon types, where each trial:

- yields a random coupon (uniformly distributed), and
- trials are independent

Let T be the number of trials needed to collect *all* n coupons. We are interested in $\mathbb{E}[T]$.

The expectation can be simply calculated using the linearity property of the expectations.

Decompose T into waiting times for new coupons:

$$T = T_1 + T_2 + \cdots + T_n$$

where:

- T_i is the time to collect i-th new coupon when $(i-1)$ are already collected, and
- each $T_i \sim \text{Geometric}(p_i)$ with $p_i = \frac{n-(i-1)}{n}$

To find $\mathbb{E}T$, we first need to calculate $\mathbb{E}T_i$ where T_i is a geometric random variable.

Lemma 2.2.
For $X \sim \text{Geom}(p)$, we have

$$\mathbb{E}[X] = \frac{1}{p}$$

First proof. By definition of expectation for discrete random variables:

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k) \\ &= \sum_{k=1}^{\infty} k(1-p)^{k-1} p \\ &= p \sum_{k=1}^{\infty} k(1-p)^{k-1}\end{aligned}$$

Let $q = 1 - p$. Recognize the series form:

$$\sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2} \quad \text{for } |q| < 1$$

Substitute back $q = 1 - p$:

$$\begin{aligned}\mathbb{E}[X] &= p \cdot \frac{1}{(1-(1-p))^2} \\ &= p \cdot \frac{1}{p^2} \\ &= \frac{1}{p}\end{aligned}$$

□

Second proof. The geometric distribution has the *memoryless property*:

$$\mathbb{E}[X] = 1 + (1-p)\mathbb{E}[X]$$

- The "1" accounts for the first trial
- With probability $1 - p$, we restart the process

Solving the equation:

$$\begin{aligned}\mathbb{E}[X] &= 1 + (1-p)\mathbb{E}[X] \\ \implies p\mathbb{E}[X] &= 1 \\ \implies \mathbb{E}[X] &= \frac{1}{p}\end{aligned}$$

□

Now we are ready to analyze the expectation of trials in the coupon collector problem.

Theorem 2.3.

$$\mathbb{E}[T] = nH_n = n \sum_{k=1}^{n} \frac{1}{k}$$

where H_n is the n-th harmonic number.

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Proof. Using linearity of expectation and geometric distribution properties:

$$\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[T_i]$$
$$\mathbb{E}[T_i] = \frac{1}{p_i} = \frac{n}{n-i+1}$$
$$\Rightarrow \mathbb{E}[T] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{k=1}^{n} \frac{1}{k} = nH_n$$

□

For large n, using harmonic series approximation:

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \cdots$$

where $\gamma \approx 0.5772$ (Euler-Mascheroni constant). Thus:

$$\mathbb{E}[T] \approx n \ln n + \gamma n + \frac{1}{2} + O\left(\frac{1}{n}\right)$$

2.3 Finding the k-th largest number

Now we apply the linearity of expectations to analyze a randomized algorithm. Given an unsorted array A of n distinct numbers and an integer $k \in \{1, \dots, n\}$, consider the problem to find the k-th largest number in A. The following QuickSelect algorithm solves the problem recursively.

Algorithm 1 Randomized Quickselect for k-th Largest Element

Require: Array A of n distinct elements, integer k where $1 \leq k \leq n$
Ensure: The k-th largest element in A

function QUICKSELECT(A, k)
  if $|A| = 1$ **then**
    return $A[0]$ ▷ Base case
  end if
  Choose pivot p uniformly at random from A
  Partition A into:
  $A_{\text{left}} = \{x \in A \mid x > p\}$ ▷ Elements > pivot
  $A_{\text{mid}} = \{x \in A \mid x = p\}$ ▷ Pivot itself
  $A_{\text{right}} = \{x \in A \mid x < p\}$ ▷ Elements < pivot
  if $|A_{\text{left}}| \geq k$ **then**
    return QUICKSELECT(A_{left}, k) ▷ Recurse on left
  else if $|A_{\text{left}}| + |A_{\text{mid}}| \geq k$ **then**
    return p ▷ Pivot is the answer
  else
    return QUICKSELECT(A_{right}, $k - |A_{\text{left}}| - |A_{\text{mid}}|$) ▷ Recurse on right
  end if
end function

Now we analyze the running time of the QuickSelect algorithm.

Partitioning Step The algorithm partitions the array into three subarrays: elements greater than, equal to, and less than the pivot. This requires a single linear scan of the array, taking $O(n)$ time.

Recursive Step The algorithm recurses on either A_{left} or A_{right}. The key observation is that the pivot is chosen *uniformly at random*. Let X be the rank of the pivot in the sorted array. Since the pivot is chosen uniformly at random, X is uniformly distributed over $\{1, 2, ..., n\}$. The algorithm recurses on:

- A_{left} if $k \le |A_{\text{left}}|$ (size $X-1$),
- A_{right} if $k > |A_{\text{left}}| + |A_{\text{mid}}|$ (size $n-X$).

The worst-case recursion occurs when the subarray size is maximized. To bound the *expected* subarray size, observe that for at least half of the pivots, X lies in the middle half of the array (i.e., $\frac{n}{4} \le X \le \frac{3n}{4}$). For these "good" pivots, the subarray size is $\le \frac{3n}{4}$. The probability of choosing a good pivot is $\frac{1}{2}$, leading to the recurrence:

$$T(n) \le \frac{1}{2} \underbrace{T\left(\frac{3n}{4}\right)}_{\text{Good pivot}} + \frac{1}{2} \underbrace{T(n)}_{\text{Bad pivot}} + \underbrace{O(n)}_{\text{Partitioning}}$$

Solving the Recurrence Unroll the recurrence:

$$T(n) \le T\left(\frac{3n}{4}\right) + cn \le T\left(\frac{9n}{16}\right) + cn + c\left(\frac{3n}{4}\right) \le \cdots$$

This forms a geometric series:

$$T(n) \le cn\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right) = cn \cdot \frac{1}{1-\frac{3}{4}} = 4cn = O(n)$$

By the linearity of expectation and geometric series convergence, the expected running time of Quickselect is $O(n)$. The random pivot ensures no persistent adversarial cases, making the average-case analysis applicable to all inputs.

An alternate analysis is to use the *law of total expectation*

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X].$$

Let X_i be the length of the subarray at the i-th recursive step. Clearly we have $X_1 = n$. Since the pivot is chosen uniformly at random, we obtain

$$\mathbb{E}[X_{i+1} \mid X_i] \le \frac{3}{4} X_i.$$

Why?
Suppose the pivot is the ℓ-largest. Then $\mathbb{E}[X_{i+1} \mid X_i] \le \max\{\ell - 1, X_i - \ell\}$.

Using the law of total expectation we have $\mathbb{E}[X_{i+1}] \le \frac{3}{4}\mathbb{E}[X_i]$. Thus the expected running time $T(n) = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = O(n)$ by the linearity of expectation.

Finally, we give an informal proof of the law of total expectation.

$$\begin{aligned}
\mathbb{E}[X] &= \int x \Pr[X = x] \, \mathrm{d}x \\
\mathbb{E}[X \mid Y = y] &= \int x \Pr[X = x \mid Y = y] \, \mathrm{d}x \\
\mathbb{E}[\mathbb{E}[X \mid Y]] &= \int \left(\int x \Pr[X = x \mid Y = y] \, \mathrm{d}x \right) \Pr[Y = y] \, \mathrm{d}y \\
&= \int \int x \Pr[X = x, Y = y] \, \mathrm{d}x \, \mathrm{d}y \\
&= \int x \left(\int \Pr[X = x, Y = y] \, \mathrm{d}y \right) \mathrm{d}x \\
&= \int x \Pr[X = x] \, \mathrm{d}x \\
&= \mathbb{E}[X].
\end{aligned}$$

2.4 Indicator Random Variable

When applying the linearity of expectation, we usually use the approach of indicator random variables. An **indicator random variable** X for an event A is defined as:

$$X = \mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

The key property is $\mathbb{E}[X] = \Pr(A)$, which often gives an easy way to calculate the expectation of indicator variable.

Example 2.4.
A monkey types randomly on a 26-letter keyboard. What is the expected number of appearance of "ALGORITHMS" (length $k = 10$) in the first 10^9 letters?
Let X_i indicates the occurrence of "ALGORITHMS" starting at position i, then $X = \sum X_i$, and clearly $\mathbb{E}[X_i] = \Pr[X_i = 1] = 26^{-10}$. So $\mathbb{E}X = \sum \mathbb{E}[X_i] = \dfrac{10^9 - 9}{26^{10}}$.

The infinite monkey theorem states that a monkey hitting keys at random on a typewriter keyboard for an infinite amount of time will almost surely type any given text, including the complete works of William Shakespeare.

Example 2.5.
Annother example is the analysis of running time of QuickSort. Let $T(n)$ be the expected times of comparison in the QuickSort algorithm. Of course we can solve the recurrence

$$T(n) = n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} T(k)$$

by generating functions. Let $G(x) = \sum_{n=0}^{\infty} T_n x^n$. Then we have

$$\sum_{n=0}^{\infty} nT_n x^n = \sum_{n=0}^{\infty} n(n-1)x^n + 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1} T_k\right)x^n,$$

which implies that $xG'(x) = \dfrac{2x^2}{(1-x)^3} + \dfrac{2x}{1-x}G(x)$. So

$$G(x) = \frac{2\ln(1-x)^{-1} - 2x + T_0}{(1-x)^2} = 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(n-k+1)\frac{1}{k} - n\right)x^n.$$

It follows that $T_n = 2\sum_{k=1}^{n}(n-k+1)\dfrac{1}{k} - 2n = 2(n+1)H_n - 4n$.
Now we apply the linearity of expectation to give a simpler analysis. Let X = total number of comparisons, and assume that the array consists of n numbers $x_1 > x_2 > \cdots > x_n$. Define $X_{ij} = \mathbf{1}_{x_i \text{ and } x_j \text{ are compared}}$. Then $X = \sum_{i<j} X_{ij}$.
The key observation is that once some element x_k between x_i and x_j is chosen to be the pivot, x_i and x_j are divided into to subarrays, and they cannot be compared. Thus, $\Pr(X_{ij} = 1) = \frac{2}{j-i+1}$ since they compared iff first pivot in $\{x_i, \ldots, x_j\}$ is x_i or x_j. Finally, applying the linearity of expectation,

$$\mathbb{E}[X] = \sum_{i<j}\frac{2}{j-i+1} = 2\sum_{k=2}^{n}(H_k - 1) = O(n\log n)$$

You may have known the median-of-median algorithm to choose pivots. But we can simply choose pivots uniformly at random, and show that the expected running time is $O(n\log n)$.

In fact, $\mathbb{E}[X]$ has a closed form:

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i<j}\frac{2}{j-i+1} = 2\sum_{k=2}^{n}(H_k - 1) \\ &= 2\sum_{k=1}^{n}(n-k+1)\frac{1}{k} - 2n \\ &= 2(n+1)H_n - 4n.\end{aligned}$$

2.5 Karp-Upfal-Wigderson Bound

While analyzing randomized algorithms involving recursive calls,if we use $T(n)$ to denote (an upper bound of) the recursive depth on instances of size n, one often meets a recurrence like:

$$T(n) \le 1 + T(n - X_n)$$

where X_n is a random variable indicating the size reduced in the recursive call. We assume we can compute a lower bound on $\mathbb{E}[X_n]$ for each n, and we want to translate this lower bound into an upper bound on $\mathbb{E}[T(n)]$. The following useful inequality due to Karp, Upfal and Wigderson[KUW88] provides an upper bound on $\mathbb{E}[T(n)]$.

Theorem 2.4. (Karp-Upfal-Wigderson Inequality)
Suppose $T : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$T(n) = \begin{cases} 1 + T(n - X_n) & n > c \\ 0 & n = c \end{cases}$$

where X_n is an integer valued random variable such that $0 \le X_n \le n - c$. If $\mathbb{E}[X_n] \ge \mu(n)$ where $\mu(n)$ is a positive nondecreasing function, then

$$\mathbb{E}T(n) \le \int_c^n \mu(x)^{-1} \, \mathrm{d}x$$

The condition $T(c) = 0$ means that when the input size is equal to or below c, our algorithm can terminate without further recursive calls. So one can imagine that we initially stand at the point $n > c$ on the real line and walk towards the point c. The instantaneous velocity at the point t is X_t, who has a lower bound $\mu(t)$ in expectation. Therefore, if everything goes on as the expectation, the total time one costs to arrive at the point c from the point n should be upper bounded by a term like $\int_c^n \mu(t)^{-1} \, \mathrm{d}t$. However, $\mu(t)$ is only a lower bound for the velocity X_t *in expectation*, and it is possible that $\mathbb{E}[1/X_t]$ is unbounded. So we cannot obtain the upper bound on time in the most straightforward way. Nevertheless, KUW inequality says that it does hold as we expect.

Proof. We prove the theorem by induction on n. If $n = c$, then $\mathbb{E}[T(c)] = 0$ and the theorem trivially holds. So we let $n > c$ and assume the theorem is true for smaller n. Now we bound $\mathbb{E}[T(n)]$ by linearity of expectation. But we should be careful with the possibility that $X_n = 0$:

$$\begin{aligned} \mathbb{E}[T(n)] &= 1 + \mathbb{E}[T(n - X_n)] \\ &= 1 + \mathbb{E}[T(n - X_n) \mid X_n = 0] \Pr[X_n = 0] + \mathbb{E}[T(n - X_n) \mid X_n > 0] \Pr[X_n > 0] \\ &= 1 + p\,\mathbb{E}[T(n)] + (1 - p)\,\mathbb{E}[T(n - X_n) \mid X_n > 0] \end{aligned}$$

where $p \triangleq \Pr[X_n = 0]$ and thus $1-p = \Pr[X_n > 0]$. It yields that

$$
\begin{aligned}
\mathbb{E}[T(n)] &= \frac{1}{1-p} + \mathbb{E}[T(n-X_n) \mid X_n > 0] \\
&= \frac{1}{1-p} + \mathbb{E}[\mathbb{E}[T(n-X_n) \mid X_n] \mid X_n > 0] \\
&\le \frac{1}{1-p} + \mathbb{E}\left[\int_c^{n-X_n} \frac{1}{\mu(x)}\,\mathrm{d}x \,\middle|\, X_n > 0\right] && \textit{(by induction)} \\
&= \frac{1}{1-p} + \int_c^{n} \frac{1}{\mu(x)}\,\mathrm{d}x - \mathbb{E}\left[\int_{n-X_n}^{n} \frac{1}{\mu(x)}\,\mathrm{d}x \,\middle|\, X_n > 0\right] \\
&\le \frac{1}{1-p} + \int_c^{n} \frac{1}{\mu(x)}\,\mathrm{d}x - \mathbb{E}\left[\int_{n-X_n}^{n} \frac{1}{\mu(n)}\,\mathrm{d}x \,\middle|\, X_n > 0\right] \\
&= \frac{1}{1-p} + \int_c^{n} \frac{1}{\mu(x)}\,\mathrm{d}x - \frac{1}{\mu(n)}\mathbb{E}[X_n \mid X_n > 0] && (\spadesuit)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mu(n) \le \mathbb{E}[X_n] &= p\mathbb{E}[X_n \mid X_n = 0] + (1-p)\mathbb{E}[X_n \mid X_n > 0] \\
&= (1-p)\mathbb{E}[X_n \mid X_n > 0]
\end{aligned}
$$

So we obtain that

$$
(\spadesuit) \le \frac{1}{1-p} + \int_c^{n} \frac{1}{\mu(x)}\,\mathrm{d}x - \frac{1}{\mu(n)}\frac{\mu(n)}{1-p} = \int_c^{n} \frac{1}{\mu(x)}\,\mathrm{d}x. \qquad \square
$$

Example 2.6.
Suppose $X \sim \text{Geom}(p)$. We apply KUW inequality to compute $\mathbb{E}[X]$. Let $T(n)$ be the number of trials before the n-th success. Then

$$
T(n) = 1 + T(n-Y)
$$

where $Y \sim \text{Ber}(p)$. So $\mathbb{E}[Y] = p$, and

$$
\mathbb{E}[T(1)] \le \int_0^1 \frac{1}{p}\,\mathrm{d}x = \frac{1}{p}
$$

by selecting $\mu(t) = 1/p$.

Example 2.7.
Revisit coupon collector. To apply KUW inequality, fix n types of coupons, and let $T(m)$ be the number of draws to collect all types when exactly m types of coupons are not collected yet. Then

$$
T(m) = 1 + T(m-X)
$$

where $X \sim \text{Ber}(m/n)$. Choosing $\mu(t) = \frac{\lceil t \rceil}{n}$, it follows that

$$
\mathbb{E}[T(n)] \le \int_{0^+}^{n} \frac{n}{\lceil x \rceil}\,\mathrm{d}x = nH_n.
$$

We cannot simply choose $\mu(t) = t/n$. Why?

Example 2.8.

Consider the recursive depth of QuickSelect. Clearly $T(n) = 1 + T(n - X_n)$ where $X_n \geq \min\{m-1, n-m\}$. Thus $\mathbb{E}[X_n] \geq \frac{n}{4}$, which gives that

$$\mathbb{E}[T(n)] \leq \int_1^n \frac{4}{x} \, \mathrm{d}x = 4 \ln n \, .$$

References

[KUW88] Richard M. Karp, Eli Upfal, and Avi Wigderson. "The complexity of parallel search". In: *Journal of Computer and System Sciences* 36.2 (1988), pp. 225–253. ISSN: 0022-0000. DOI: https://doi.org/10.1016/0022-0000(88)90027-X. URL: https://www.sciencedirect.com/science/article/pii/002200008890027X.