Fundamental Relationship between Node Density and Delay in Wireless Networks with Unreliable Links

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Abstract—We investigate the fundamental relationship between node density and transmission delay in Large-scale wireless networks with unreliable links from percolation perspective. Previous works[7][2][6] have already showed the relationship between transmission delay and distance from source to destination. However, it still remains as an open question how transmission delay varies in accordance with node density. In this paper, we study the impact of node density λ on the ratio of delay and distance, denoted by $\gamma(\lambda)$. We analytically characterize the properties of $\gamma(\lambda)$ as a function of λ. And then we present upper and lower bounds to $\gamma(\lambda)$. Next, we take propagation delay into consideration and obtain further results on the upper and lower bounds of $\gamma(\lambda)$. Finally, we make simulations to verify our theoretical analysis.

Index Terms—Connectivity, Delay, Density

I. INTRODUCTION

Wireless communication sees an explosive growth in the number of customers in the past few decades, making Large-scale wireless network an important part of modern life. To ensure the successful communication between node pairs in a wireless network, the network needs to maintain full connectivity[3]. However, it is overly power consuming to achieve full connectivity in Large-scale networks(*i.e.*, the power required to maintain full connectivity increases with the size of the network). Thus, it is necessary to introduce a slightly weaker connectivity criterion, *i.e.*, an infinite connected component containing a high fraction of the network nodes exists in a network. Thanks to percolation theory[5][16], it is possible to achieve this weaker connectivity in Large-scale networks with power bounded.

Percolation theory[5], especially continuum percolation, has become a useful mathematical tool when analyzing the capacity and the connectivity of wireless networks. The most general model in Continuum Percolation, Random Connection Model(RCM), describes the behavior of connected clusters in a random geometric graph in which nodes are distributed according to poisson point process with node density λ, and two nodes share a link according to a connection function $h(r)$. A fundamental result of RCM points out a phase transition effect[1]. For $\lambda > \lambda_c$(supercritical), there exist a unique con-

[1]The condition for the existence of phase transition is $0 < \int_{\mathbb{R}^2} h(r)dr < \infty$.

nected component containing an infinite number of nodes(we also say the network is percolated). For $\lambda < \lambda_c$(subcritical), all connected component in the network are finite almost surely.

Applying percolation theory to wireless networks with unreliable links, we introduce two important concepts, *i.e.*, instantaneous connectivity and long-term connectivity. Instantaneous connectivity requires wireless network percolated all the time. Long-term connectivity requires wireless network percolated in the long run(we will elaborate it more clearly later in sectionII-C). The instantaneous critical density, denoted by λ_I, is the critical density for instantaneous connectivity and the long-term critical density, denoted by λ_L, is the critical density for long-term connectivity. Long-term connectivity is a weaker criterion for connectivity, thus $\lambda_L < \lambda_I$. The Prerequisite for communication in wireless networks is connectivity, so we only focus on the case $\lambda > \lambda_L$.

In wireless networks with unreliable links, delay is composed of two parts, the waiting delay and the propagation delay. The waiting delay is caused by the lack of instantaneous connectivity. Information cannot be transmitted to a distant destination instantaneously since the connected component is finite almost surely. It must wait for some time until some communication links are established and can transmit forward again. As for the propagation delay, it only depends on the channel condition and communication medium. It has no relationship with networks' node density λ. For ease of analysis, we first ignore the impact of propagation delay and will consider its effect in the last.

Previous works [7][2][6] have showed that if $\lambda_L < \lambda < \lambda_I$, the transmission delay scales linearly with distance between source and destination($\gamma(\lambda) > 0$), and if $\lambda > \lambda_I$, the transmission delay scales sub-linearly with distance($\gamma(\lambda) = 0$). This also indicates that delay must have some relationship with node density λ. However, what is the exact relationship between the delay-distance ratio $\gamma(\lambda)$ and the node density λ? Do there exist lower and upper bounds to this ratio $\gamma(\lambda)$? Answering these questions can help reveal the essence of delay in wireless networks with unreliable links.

In this paper, we give a more precise description of the delay in wireless networks with unreliable links. We present 3 properties of $\gamma(\lambda)$ as a function of λ. Using coupling techniques, we prove that $\gamma(\lambda)$ is a monotone decreasing

function.

And then, we come to the upper and lower bounds of $\gamma(\lambda)$, ignoring the propagation delay. For the upper bound, we first find a path between two nodes. And then we calculate the number of hops along this path and the delay at each hop. We obtain the result on upper bound through multiplication of the above two items. For the lower bound, we first introduce a concept called cluster to cluster transmission process and establish the relationship between delay and the cluster to cluster transmission process, which reveals the essence of delay in networks. Then, using the definition of delay of a cluster to cluster transmission, we obtain a lower bound of $\gamma(\lambda)$.

Next, we take propagation delay into consideration and reformulate $\gamma(\lambda)$ in this case. Propagation delay increases the delay in Large-Scale Networks, making $\gamma(\lambda) > 0$ even when $\lambda > \lambda_I$. Using similar methods, we present new upper and lower bounds to $\gamma(\lambda)$ for all $\lambda > \lambda_L$.

Finally, we make enormous simulation and further verifies our theoretical results.

The original contributions that we have made in the paper are highlighted as follows:

- We present three properties to $\gamma(\lambda)$, *i.e.*, $\gamma(\lambda)$ is uniformly bounded; $\gamma(\lambda) = 0$ whenever $\lambda > \lambda_I$; $\gamma(\lambda)$ is a monotone decreasing function.
- Ignoring propagation delay, we provide the upper bound and the lower bound to reflect the range of variation on $\gamma(\lambda)$, *i.e.*, $\frac{1}{E(S_g(\lambda)+r_0)} \leq \gamma(\lambda) \leq \inf_{\lambda' \in [\lambda_L, \lambda]} \kappa \sqrt{\frac{\lambda'}{\lambda_L}} \left(\frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)} - 1 \right).$
- Taking propagation delay into consideration, we proved that $\frac{1}{E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})+r_0} \leq \gamma(\lambda) \leq \inf_{\lambda' \in [\lambda_L, \lambda]} \frac{\kappa \sqrt{\lambda'/\lambda_L}}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)}$
  [2]
- We conduct simulations to obtain experimental values of $\gamma(\lambda)$ in the above two cases. A new observation arises from our comparison between theoretical and simulation results is that the delay-distance ratio $\gamma(\lambda)$ can be estimated by the lower bound in relative dense networks while the experimental values of $\gamma(\lambda)$ get closer to the upper bound as λ decreases. This also justifies the soundness of our theoretical conclusion.

The rest of the paper is organized as follows. In section II, we introduce our network model, several useful mathematical tools and some important notations. In section III, we first give three properties of $\gamma(\lambda)$, and then present our main results concerning the upper and lower bound of $\gamma(\lambda)$. The analysis process to obtain the upper and lower bounds is given in section IV. Simulation results are presented in Section V to support our theoretical findings. We summarize the paper in Section VI. Some proofs of the theorems and lemmas are presented in line or in Appendix.

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[2] κ is a constant independent of λ; $S_g(\lambda)$ is a random variable(see section II-E for its definition).

II. NETWORK MODEL

In this section, we present the network model in this paper. First, we list some properties of poisson point process that are frequently used in this paper. Then we give a brief description of Random Connection Model. Next, random geometric graph is introduced including some important concepts. After that, we present the first passage percolation model and give the accurate definition of delay-distance ratio $\gamma(\lambda)$. Finally, we list some important notations in this paper.

A. Poisson Point Process

In large-scale wireless networks, nodes are distributed according to Poisson Point Process. In the following analysis, we will frequently use the following two classical results on Poisson Point Process.

Lemma 1. *[14]Let Γ be a potentially inhomogeneous Poisson process on $\mathbb{R}^d$ with density function $\lambda(x)$, where $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$. Suppose that we obtain Γ' by independently coloring points $x \in \Gamma$ according to probabilities $p(x)$. Then Γ' and $\Gamma - \Gamma'$ are two independent Poisson processes with density function $p(x)\lambda(x)$ and $(1 - p(x))\lambda(x)$, respectively.*

Lemma 2. *Let Γ, Γ' be two independent inhomogeneous Poisson process on $\mathbb{R}^d$ with density functions $\lambda(x)$ and $\lambda'(x)$, respectively, where $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$. Suppose that we obtain $\Gamma + \Gamma'$ by superposing Γ' on Γ. Then $\Gamma + \Gamma'$ is a Poisson processes with density function $\lambda(x) + \lambda'(x)$.*

The above two lemmas point out that the decomposition or superposition of Poisson Point Processes remains to be Poisson Point Process. This is very useful in coupling techniques.

B. Random Connection Model

Random Connection Model(RCM) is the most general model in Continuum Percolation Theory. In Random Connection Model, nodes are distributed according to Poisson point process [13] in $\mathbb{R}^d$. Here we only focus on the case of $\mathbb{R}^2$ with node density $\lambda > 0$. Each node x connect to another node y according to the connection function $h(r)$, where r is the distance between x and y, and $h(r)$ satisfy $0 < \int_{\mathbb{R}^2} h(r)dr < \infty$.

We denote the Random Connection Model by $\mathcal{G}(\lambda, r_0, h(r))$. Then $\mathcal{G}(\lambda, r_0, h(r))$ is a set of nodes connected by random links. For convenient, we assume the origin $0 \in \mathcal{G}(\lambda, r_0, h(r))$.

Obviously, $\mathcal{G}(\lambda, r_0, h(r))$ is composed of one or several disjointed connected clusters. Let us denote $W(A), A \subseteq \mathcal{G}(\lambda, r_0, h(r))$, the set of nodes attainable from nodes in set A, *i.e.*,

$$W(A) = \{x \in \mathcal{G}(\lambda, r_0, h(r)) | \exists a \in A, a \leftrightarrow x\},$$

where, $a \leftrightarrow x$ means that nodes a and x are in the same connected component.

Besides, we use $|W|$ to represent the cardinality of set W. And we write $\theta_h(\lambda) = \mathbb{P}_{\lambda,h}(|W(\{0\})| = \infty)$ and $\chi_h(\lambda) = E_{\lambda,h}(|W(\{0\})|)$[3].

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[3] $E(x)$ is the expectation of random variable x

Then, the critical density can be determined in two ways, *i.e.*,

$$\lambda_\theta(h) = \inf\{\lambda|\theta_h(\lambda) > 0\}; \tag{1}$$

$$\lambda_\chi(h) = \inf\{\lambda|\chi_h(\lambda) = \infty\}. \tag{2}$$

According to Theorem 6.2 in [16], $\lambda_\theta(h) = \lambda_\chi(h) = \lambda_c(h)$. Furthermore, $0 < \lambda_c(h) < \infty$. And there exists a unique infinite connected cluster if $\lambda > \lambda_c(h)$ (supercritical). This infinite connected cluster is also called the giant component, denoted by $\mathcal{C}(\mathcal{G}(\lambda, r_0, h(r)))$. On the other hand, if $\lambda < \lambda_c(h)$(subcritical), all the connected components are finite almost surely.

Random Connection Model is just one kind of continuum percolation model. Another continuum percolation model is Poisson Boolean Model $\mathcal{B}(\lambda, r)$. In Poisson Boolean Model $\mathcal{B}(\lambda, r)$, nodes are distributed according t Poisson Point Process with density λ, and two nodes can communicate if and only if their distance is smaller than r. Poisson Boolean Model can also be seen as a collection of discs with radius $\frac{r}{2}$. Poisson Boolean Model is a special case of Random Connection Model, thus the conclusions for Random Connection Model still hold for Poisson Boolean Model.

C. Random Geometric Model

Assume nodes are distributed according to Poisson Point Process with node density λ in an infinite two-dimensional space $\mathbb{R}^2$. For each node u, we use u to represent both this node and its location without causing confusion. We say two nodes share a link if and only if their distance is smaller than r_0. However, due to the severe natural hazards, enemy attack or energy depletion, each link suffers the possibility to fail. We model this failure as each link opening or closing intermittently.
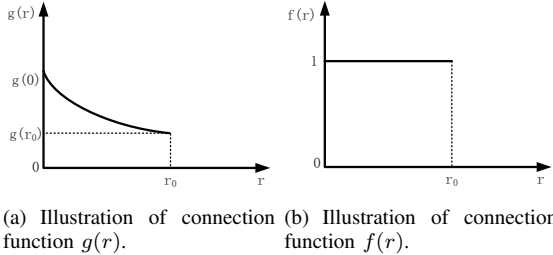

(a) Illustration of connection function $g(r)$. (b) Illustration of connection function $f(r)$.

Fig. 1.   Illustration of two connection functions.

Assume time is slotted. Consider a link with length r, at time slot t, we let it open with probability $g(r)$(Fig. 1), independent of its former states. In reality, the farther two nodes are apart, the more difficult for a successful communication. Moveover, when $r > r_0$, there exists no link. Thus, it is reasonable to assume that $g(r)$ is a monotone decreasing function and $g(r) = 0$ whenever $r > r_0$. Besides, we place another restriction on $g(r)$, *i.e.*,

$$1 > g(0) \geq g(r) \geq g(r_0) > 0, 0 \leq r \leq r_0. \tag{3}$$

Then the network at each time slot t can be represented by a Random Connection Model $\mathcal{G}_t(\lambda, r_0, g(r))$. Here, we use subscript t to indicate that the network is dynamic. Note that if $\lambda > \lambda_c(g(r))$, $\mathcal{G}_t(\lambda, r_0, g(r))$ is percolated for all t(we also say the network has instantaneous connectivity); while if $\lambda < \lambda_c(g(r))$, $\mathcal{G}_t(\lambda, r_0, g(r))$ is not percolated for all t. Thus, the instantaneous critical density $\lambda_I = \lambda_c(g(r))$.

Next, we introduce the concept of long-term connectivity. We first construct a new geometric graph. The location of all nodes in this graph is the same as that in $\mathcal{G}_t(\lambda, r_0, g(r))$. Two nodes x and y share a link in this graph if and only if there exist t, such that x and y share an open link in $\mathcal{G}_t(\lambda, r_0, g(r))$. Note that x and y has the potential to share a link in $\mathcal{G}_t(\lambda, r_0, g(r))$ for some t, whenever their distance is smaller r_0. Thus this new geometric graph can be represented by a Random Connection Model $\mathcal{G}(\lambda, r_0, f(r))$(it can be also represented by Poisson Boolean Model $\mathcal{B}(\lambda, r_0)$). Here, $f(r) = 1$ when $r < r_0$, and $f(r) = 0$ when $r > r_0$(Fig. 1). We say the wireless network has long-term connectivity if and only if $\mathcal{G}(\lambda, r_0, f(r))$ is percolated. And the critical density $\lambda_L = \lambda_c(f(r))$ is defined as the long-term critical density.

As for the instantaneous critical density and the long-term critical density, we have the following relationship.

Lemma 3. $\lambda_L \leq \lambda_I$.

Proof: For any $\lambda > \lambda_I$, there exist one infinite connected cluster in $\mathcal{G}_t(\lambda, r_0, g(r))$. From the definition of $\mathcal{G}(\lambda, r_0, f(r))$, we have

$$\mathcal{G}(\lambda, r_0, f(r)) = \cup_{t=1}^{\infty} \mathcal{G}_t(\lambda, r_0, g(r)).$$

Thus, there must exist one infinite connected cluster in $\mathcal{G}(\lambda, r_0, f(r))$. Therefore, $\lambda > \lambda_L$. Hence, $\lambda_L \leq \lambda_I$. ∎

Since the prerequisite for communication in large-scale wireless network is connectivity, it is enough to only focus on the case $\lambda > \lambda_L$.

D. First Passage Percolation Model

This paper is based on the First Passage Percolation Model. First Passage Percolation, first formulated by Hammersley and Welsh[1] in 1965, can be a very powerful tool for analysis of transmission delay in Large-scale networks.

Given a Random Connection Model $\mathcal{G}(\lambda, r_0, h(r))$, attach each link e of $\mathcal{G}(\lambda, r_0, h(r))$ a random variable $T_c(e)$, representing the time needed to pass through the link e. Consider a path π, the passage time is defined as

$$T_p(\pi) = \sum_{e \in \pi}(T_c(e)).$$

And for any two nodes x and y(x, y are not necessarily adjacent), the first-passage time $T_\lambda(x, y)$ is given by

$$T_\lambda(x, y) = \inf\{T_p(\pi) : \pi \text{ is a pass from } x \text{ to } y\}.$$

In this paper, the states of links are dynamic. Sometimes, information must wait at one end of a link until this link is on. This is equivalent to introducing a crossing time $T_c(e)$ to each link $e \in \mathcal{G}(\lambda, r_0, f(r))$. Assume that the length of the link e

is $0 < r < r_0$, $T_c(e)$ satisfy the Geometric distribution(here we have ignored the propagation delay),*i.e.*,

$$\mathbb{P}(T_c(e) = k) = (1 - g(r))^k g(r). \tag{4}$$

Eqn. (3) assures that $0 < E(T_c(e)) < \infty$. Using Liggett's subadditive ergodic theorem[20], previous works have proved that, for $x, y \in \mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$, when $\lambda > \lambda_L$,

$$\lim_{d(x,y) \to \infty} \frac{T_\lambda(x, y)}{d(x, y)} = \gamma(\lambda). \tag{5}$$

Mooveover, if $\lambda_L < \lambda < \lambda_I$, $\gamma > 0$; while if $\lambda > \lambda_I$, $\gamma = 0$.

Eqn. (5) is also the definition of $\gamma(\lambda)$. From the former result, we can also see that γ must depend on λ. However, the existing results only point out when $\gamma(\lambda)$ equals to 0, and when it is larger than 0. The exact relationship between $\gamma(\lambda)$ and λ still remains as an open question. In this paper, we will give a more precise description on $\gamma(\lambda)$.

E. Useful Notations

Some useful notations are listed as follows.

- (Section II-B)$\mathcal{G}(\lambda, r_0, h)$ is a Random Connection Model, and h is the connection function; $\mathcal{B}(\lambda, r)$ is the Poisson Boolean Model; we use $\mathcal{C}(\mathcal{G}(\lambda, r_0, h))(\mathcal{C}(\mathcal{B}(\lambda, r)))$ to represent the giant component of $\mathcal{G}(\lambda, r_0, h)(\mathcal{B}(\lambda, r))$.
- (Section II-C) $\mathcal{G}_t(\lambda, r_0, g)$ is the instantaneous geometric graph at time slot t and its critical density is λ_I; $\mathcal{G}(\lambda, r_0, f)$ is the long-term geometric graph and its critical density is λ_L.
- $\mathbb{P}(\bullet)$ represents the probability of some event;$E(\bullet)$ represents the expectation of a random variable;$z_x(z_y)$ represents the $x(y)$-coordinate of z; $d(u, v) = \parallel u - v \parallel$ is the Euclidean distance between node u and v.
- (Section IV-C)$H(z_0, a)$ is a circular region defined as $H(z_0, a) = \{z = (z_x, z_y) \in \mathbb{R}^2 | \parallel z - z_0 \parallel < a\}$. The random variable $S_{g,t,u}(\lambda)$ is defined as $S_{g,t,u}(\lambda) = \sup\{a| \exists$ node $v \in H(u, a), v \leftrightarrow u$ at time slot $t\}$. Actually, $S_{g,t,u}(\lambda)$ is independent of t and u. Thus, we write $S_{g,t,u}(\lambda)$ as $S_g(\lambda)$ for short.
- (Section II-D) $T_c(e)$ is the passage time for a link e; $T_p(\pi)$ is the passage time for a path π; $T_\lambda(x, y)$ is the first passage time from node x to y; (Section IV-B) $T_p(\Pi)$ is the passage time for a cluster to cluster transmission process Π;
- (Section IV-A)$N_\lambda(d(u, v))$ is the minimum number of hops from node u to v.
- π represents a path; Π represents a cluster to cluster transmission process.

III. MAIN RESULTS

In this section, we first give some properties on the delay-distance ratio $\gamma(\lambda)$. And then we present our main results concerning the tradeoff between node density and $\gamma(\lambda)$ in wireless networks with unreliable links, in which an upper bound and a lower bound for $\gamma(\lambda)$, are given.

A. Properties of $\gamma(\lambda)$

$\gamma(\lambda)$ can be seen as a function mapping from $[\lambda_L, \infty)$ to $\mathbb{R}$. The properties of $\gamma(\lambda)$ are listed below.

Theorem 1. $\gamma(\lambda)$ *has the following three properties:*

- *there exists $\gamma_M < \infty$, such that for $\forall \lambda$, $\gamma(\lambda) \leq \gamma_M$;*
- *for $\forall \lambda > \lambda_I$, $\gamma(\lambda) = 0$;*
- *$\gamma(\lambda)$ is a monotone decreasing function.*

Proof: The first property can be proved later, so we do not elaborate it here. The second property has already been proved by previous literatures[7] [6][2][21]. Thus, we only present the proof of property 3 here.

Given $\lambda_1 > \lambda_2$, consider two Random Connection Models $\mathcal{G}_t(\lambda_1, r_0, g(r))$ and $\mathcal{G}_t(\lambda_2, r_0, g(r))$. We use coupling technique to prove $\gamma(\lambda_1) \leq \gamma(\lambda_2)$. Nodes in $\mathcal{G}_t(\lambda_1, r_0, g(r))$ and $\mathcal{G}_t(\lambda_2, r_0, g(r))$ are distributed according to Poisson Point Process Γ_1 and Γ_2 with node densities λ_1 and λ_2, respectively. According to lemma 2, Γ_1 can be seen as the superposition of Γ_2 and another Poisson Point Process Γ' with node density $\lambda_2 - \lambda_1$.

Consider nodes $x, y \in \Gamma_2$, since $\Gamma_2 \subseteq \Gamma_1$, we obtain $x, y \in \Gamma_1$. For any path π connecting x and y in $\mathcal{G}_t(\lambda_2, r_0, g(r))$, this path also exists in $\mathcal{G}_t(\lambda_1, r_0, g(r))$. And the delay from x to y, $T_\lambda(x, y)$, is defined as the minimum delay among all paths connecting x and y. Thus,

$$T_{\lambda_1}(x, y) \leq T_{\lambda_2}(x, y).$$

Divide the above inequality by $d(x, y)$, and let $d(x, y) \to \infty$, we obtain

$$\gamma(\lambda_1) \leq \gamma(\lambda_2).$$

∎

We also have a conjecture about $\gamma(\lambda)$, *i.e.*, $\gamma(\lambda)$ is a continuous function. However, we fail to prove it.

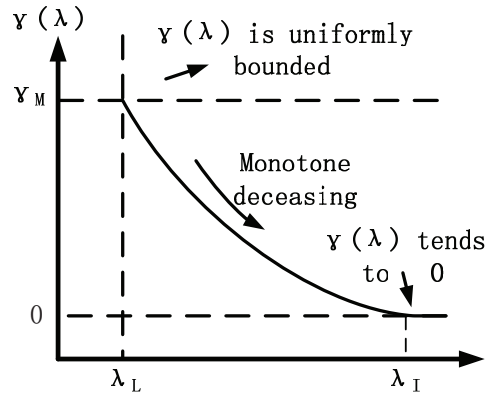According to Theorem 1, we can sketch $\gamma(\lambda)$ out(Fig. 2).

Fig. 2. Sketch of $\gamma(\lambda)$.

B. Main results on $\gamma(\lambda)$

We have obtained several properties of $\gamma(\lambda)$. Now we are ready to present our main results.

Theorem 2. *Given Random Connection Model* $\mathcal{G}_t(\lambda, r_0, g(r))$ *with* $\lambda_L < \lambda < \lambda_I$, *the corresponding* $\gamma(\lambda)$ *satisfies*

$$\frac{1}{E(S_g(\lambda) + r_0)} \leq \gamma(\lambda) \leq \inf_{\lambda' \in [\lambda_L, \lambda]} \kappa \sqrt{\frac{\lambda'}{\lambda_L}} \left(\frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)} - 1 \right), \tag{6}$$

where κ *is a constant independent of* λ.

If we take propagation delay into consideration, we have the following results.

Theorem 3. *Given Random Connection Model* $\mathcal{G}_t(\lambda, r_0, g(r))$ *with* $\lambda > \lambda_L$, τ *is the propagation delay for a existing link and* $\tau < 1$. *Then the corresponding* $\gamma(\lambda)$ *satisfies*

$$\frac{1}{E(\min\{S_g(\lambda), \frac{r_0}{\tau}\}) + r_0} \leq \gamma(\lambda) \leq \inf_{\lambda' \in [\lambda_L, \lambda]} \frac{\kappa \sqrt{\lambda'/\lambda_L}}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)}, \tag{7}$$

where κ *is a constant independent of* λ.

Our results provide a theoretical way to estimate delay in Large-scale wireless networks. We use connection function $g(r)$ to represent the condition of a Large-scale wireless network, making our results applicable to most cases in real networks. Also, waiting delay and propagation delay are both taken into account in our formulation, making our results more reliable.

IV. Upper and Lower bounds of $\gamma(\lambda)$

In this section, we first give an upper bound to the delay-distance ratio, $\gamma(\lambda)$. And then, we make further analysis on transmission delay and introduce a concept called *cluster to cluster* transmission. Using this concept, we derive another upper bound and a lower bound. Finally, we take propagation delay into consideration, and formulate its impact on $\gamma(\lambda)$.

Turn back to the definition of $\gamma(\lambda)$(Eqn. 5).

$$\gamma(\lambda) = \lim_{d(x,y) \to \infty} \frac{T_\lambda(x, y)}{d(x, y)},$$

where x, y belongs to the giant component of $\mathcal{G}(\lambda, r_0, f(r))$. However, we needn't calculate $\gamma(\lambda)$ for all $x, y \in \mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$. The correctness of this assertion is assured by the following lemma.

Lemma 4. *Given a convergent sequence* $\{x_k\}, k = 1, 2, ...,$ *and* $\lim_{k \to \infty} x_k = x_0$. $\{y_k\}, k = 1, 2, ...,$ *is a subsequence of* $\{x_k\}$, *and* $\lim_{k \to \infty} y_k = y_0$. *Then* $x_0 = y_0$.

Obviously, the number of nodes in $\mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$ is countable. We enumerate for all nodes. We randomly select a node and label it as x_0, and then label other nodes according to the distance from x_0(larger subscript means larger distance from x_0). Define sequence $\{m_k, k = 1, 2, ..., \}, m_k =$

$\frac{T_\lambda(x_0, x_k)}{d(x_0, x_k)}$, then $\lim_{k \to \infty} m_k = \gamma(\lambda)$. According to lemma 4, we only need to find a subset of nodes of $\mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$(the cardinality of this subset must be infinity), and calculate $\gamma(\lambda)$ from this subset. This technique is used in deriving the upper bound.

A. Upper Bound of $\gamma(\lambda)$

The definition of of delay is defined as the minimum delay along all paths connecting source and destination nodes. Thus, it must be smaller than or equal to the delay along one path. In this part, we will first find a subset of nodes of $\mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$. And then we find a path for each node pair in this subset. After that, we calculate the delay along this path. Finally, dividing the delay by distance, we obtain an upper bound of $\gamma(\lambda)$.

Before proceeding, we need the following lemma.

Lemma 5. *Consider Poisson Boolean models in* $\mathbb{R}^2$. *Let* $\lambda_c(r)$ *denote the critical density in the case where the transmission range is* r. *Then it is the case that*

$$\lambda_c(r_1)r_1^2 = \lambda_c(r_2)r_2^2,$$

where $r_1, r_2 > 0$.

Proof: See Proposition 2.10 in [16]. ∎

In long-term critical density λ_L is also the critical density of Poisson Boolean Model with transmission range r_0. Consider the network with density $\lambda > \lambda_L$, according to lemma 5, we immediately know that when $\lambda r^2 > \lambda_L r_0^2$, *i.e.*, $r > \sqrt{\frac{\lambda_L}{\lambda}} \cdot r_0$, Poisson Boolean Model $\mathscr{B}(\lambda, r)$ is percolated.

Let $0 < \varepsilon < \sqrt{\frac{\lambda}{\lambda_L}} - 1$ and $\tilde{r} = r_0 \sqrt{\frac{\lambda_L}{\lambda}}(1 + \epsilon)$, then $\mathcal{B}(\lambda, \tilde{r})$ is percolated. Note that, in Random Connection Model $\mathcal{G}(\lambda, r_0, f(r))$, the transmission is $r_0 > \tilde{r}$. Thus, $\mathcal{B}(\lambda, \tilde{r})$ is a subgraph of $\mathcal{G}(\lambda, r_0, f(r))$. We denote the giant component of $\mathcal{B}(\lambda, \tilde{r})$ by $\mathcal{C}(\mathcal{B}(\lambda, \tilde{r}))$. According to the uniqueness of giant component in supercritical case, there must be $\mathcal{C}(\mathcal{B}(\lambda, \tilde{r})) \subseteq \mathcal{C}(\mathcal{G}(\lambda, r_0, f(r)))$. According to lemma 4, when calculating $\gamma(\lambda)$, we only need to focus on the case that both nodes belong to $\mathcal{C}(\mathcal{B}(\lambda, \tilde{r}))$.

Assume that nodes $u, v \in \mathcal{C}(\mathcal{B}(\lambda, \tilde{r}))$. Then there exists at least one path in $\mathcal{B}(\lambda, \tilde{r})$ from u to v. We choose the path with minimum number of hops, and denote it by π_m.

Up to now, we have found a path connecting u and v. Next, we are to calculate the delay along this path. To start with, we need to work out the number of hops, denoted by $N_\lambda(d(u, v))$, in π_m. We needn't calculate each $N_\lambda(d(u, v))$ for different λ. We can find the relationship of $N_\lambda(d(u, v))$ for different λ in the following way.

Scale the network up by $\sqrt{\frac{\lambda}{\lambda_L}}$, and then the node density becomes λ_L, the transmission range becomes $(1 + \epsilon)r_0$, and the distance between u and v becomes $d(u, v)\sqrt{\frac{\lambda}{\lambda_L}}$. Then it is equivalent to calculate $N_{\lambda_L}(d(u, v)\sqrt{\frac{\lambda}{\lambda_L}})$. Next, we present the lemma concerning $N_{\lambda_L}(d)$.

Lemma 6. *Given $\mathcal{B}(\lambda_L, (1+\epsilon)r_0)$, and $u, v \in \mathcal{C}(\mathcal{B}(\lambda_L, 1+\epsilon))$, the minimal number of hops needed for transmitting information from u to v is $N_{\lambda_L}(d(u,v))$. Then there exist κ such that*

$$\lim_{d(u,v)\to\infty} \frac{N_{\lambda_L}(d(u,v))}{d(u,v)} = \kappa.$$

The proof of this lemma 6 is based on a conclusion on subadditivity and is given in Appendix I.

Lemma 7. *(Liggett [20])Let $\{S_{l,m}\}$ be a collection of random variables indexed by integers $0 \leq l \leq m$. Suppose $\{S_{l,m}\}$ has the following properties:*

1) *$S_{0,m} \leq S_{0,l} + S_{l,m}, 0 \leq l \leq m$;*
2) *$\{S_{(m-1)k,mk}, m \geq 1\}$ is a stationary process for each k;*
3) *$\{S_{l,l+k}, k \geq 0\} = \{S_{l+1,l+k+1}, k \geq 0\}$ in distribution for each l;*
4) *$E[|S_{0,m}|] < \infty$ for each m.*

Then $\alpha \triangleq \lim_{m\to\infty} \frac{E[S_{0,m}]}{m} = \inf_{m\geq 1} \frac{E[S_{0,m}]}{m}$; $S \triangleq \lim_{m\to\infty} \frac{S_{0,m}}{m}$ exists with probability 1 and $E[S] = \alpha$. Furthermore, if

5. *the stationary process $\{S_{(m-1)k,mk}, m \geq 1\}$ is ergodic;*

then $S = \alpha$ with probability 1.

According to lemma 6, we immediately get

$$
\begin{aligned}
\lim_{d(u,v)\to\infty} \frac{N_\lambda(d(u,v))}{d(u,v)} &= \lim_{d(u,v)\to\infty} \frac{N_{\lambda_L}(d(u,v)\sqrt{\frac{\lambda}{\lambda_L}})}{d(u,v)} \\
&= \kappa\sqrt{\frac{\lambda}{\lambda_L}}.
\end{aligned}
\tag{8}
$$

Then we calculate the delay $T_p(\pi_m)$ along path π_m. According to Strong Large Number Theory, with high probability, we have

$$T_p(\pi_m) = \sum_{e\in\pi_m} T_c(e) = N_\lambda(d)E[T_c(e)].$$

Therefore,

$$
\begin{aligned}
\gamma(\lambda) &= \lim_{d\to\infty} \frac{T_\lambda(u,v)}{d(u,v)} \\
&\leq \lim_{d\to\infty} \frac{T_p(\pi_m)}{d(u,v)} \\
&= E[T_c(e)] \lim_{d\to\infty} \frac{N_\lambda(d(u,v))}{d(u,v)} \\
&= \kappa\sqrt{\frac{\lambda}{\lambda_L}} E[T_c(e)].
\end{aligned}
\tag{9}
$$

From the definition of the path π_m, we know that the length of each hop is smaller than $\tilde{r} = r_0\sqrt{\frac{\lambda_L}{\lambda}}(1+\epsilon)$. Besides, the connection function $g(r)$ is monotone decreasing. Thus, for a

link e' whose length is $\tilde{r}$, there must be

$$
\begin{aligned}
E[T_c(e)] &\leq E[T_c(e')] \\
&= \sum_{k=0}^{\infty} k\mathbb{P}(T_c(e') = k) \\
&= \sum_{k=0}^{\infty} k(1 - g(\tilde{r}))^k g(\tilde{r}) \\
&= \frac{1}{g(\tilde{r})} - 1.
\end{aligned}
\tag{10}
$$

Thus,

$$
\begin{aligned}
\gamma(\lambda) &= \kappa\sqrt{\frac{\lambda}{\lambda_L}} E[T_c(e)] \\
&\leq \kappa\sqrt{\frac{\lambda}{\lambda_L}}\left(\frac{1}{g(\tilde{r})} - 1\right) \\
&= \kappa\sqrt{\frac{\lambda}{\lambda_L}}\left(\frac{1}{g\left(r_0\sqrt{\frac{\lambda_L}{\lambda}}(1+\epsilon)\right)} - 1\right).
\end{aligned}
$$

Let $\epsilon \to 0$, we obtain

$$\gamma(\lambda) \leq \kappa\sqrt{\frac{\lambda}{\lambda_L}}\left(\frac{1}{g\left(r_0\sqrt{\frac{\lambda_L}{\lambda}}\right)} - 1\right).$$

Furthermore, from property 3 of theorem 1, we know $\gamma(\lambda)$ is a monotone decreasing function. Thus,

$$\gamma(\lambda) \leq \inf_{\lambda'\in[\lambda_L,\lambda]} \kappa\sqrt{\frac{\lambda'}{\lambda_L}}\left(\frac{1}{g\left(r_0\sqrt{\frac{\lambda_L}{\lambda'}}\right)} - 1\right).$$

B. Cluster to Cluster Transmission

In section IV-A, we have obtained the upper bound of $\gamma(\lambda)$ by calculating the delay along one path. However, the method used in section IV-A cannot be generalized to study the lower bound of $\gamma(\lambda)$. We don't know the number of paths connecting two nodes, nor do we know the delay along each path. Thus, it is impossible to find the minimum delay along all the paths.

To find a lower bound of $\gamma(\lambda)$, we need to make clear that when delay is introduced to network.

Consider transmitting information from node u to v. Assume that at time slot t_1, node $u_1(u_1 = u)$ transmits information to other nodes. Since we have ignored the propagation delay, all nodes connected to u_1 in geometric graph $\mathcal{G}_{t_1}(\lambda, r_0, g(r))$, denoted by w_1, receive the information instantaneously. Then, the transmission process stops. The transmission process will restart at time slot $t_2 > t_1$[4], when at least one node in w_1 find the opportunity to forward the information to a new node, denoted by u_2. At this time slot, u_2 transmits information to those nodes which are connected to u_2 in geometric graph $\mathcal{G}_{t_2}(\lambda, r_0, g(r))$, and do not belong to w_1, denoted by w_2, instantaneously. This process goes on,

---

[4]Here, we do not require t_2 to be the smallest, *i.e.*, there may exist $t_1 < t' < t_2$, such that at least one node in w_1 have the opportunity to forward the information to a new node at time slot t'.

until at time slot t_M, node u_M and the destination node v are in the same connected cluster and information is transmitted to node v instantaneously.

We can see that the cluster to cluster transmission as a series of outbursts. During each outburst, some new nodes receive the information. $w_k, k = 1, 2, ..., M$ is the set of nodes which receive the information right at the kth outburst. A cluster to cluster transmission process can be represented by $\Pi = \{(t_1, u_1), (t_2, u_2), ..., (t_M, u_M)\}$. Information can be transmitted from u to v through Π. Define the passage time for the cluster to cluster transmission process Π as

$$T_p(\Pi) = t_M - t_1.$$

Then, we have the following lemma.

Lemma 8. *Given nodes* $u, v \in \mathcal{C}((G)(\lambda, r_0, f(r)))$, *the first passage time*

$$T_\lambda(u, v) = \inf\{T_p(\Pi)|\Pi \text{ is a cluster to cluster} \\ \text{transmission process from } u \text{ to } v\}. \quad (11)$$

Proof: For convenience, we use $\mathscr{L}$ to denote the set of cluster to cluster transmission process from u to v. Then Eqn.(11) can be rewritten as

$$T_\lambda(u, v) = \inf\{T_p(\Pi)|\Pi \in \mathscr{L}\}.$$

It is easy to see that for each cluster to cluster transmission process Π from u to v,

$$T_p(\Pi) \geq T_\lambda(u, v).$$

Thus,

$$\inf\{T_p(\Pi)|\Pi \in \mathscr{L}\} \geq T_\lambda(u, v). \quad (12)$$

Next, we show that

$$\inf\{T_p(\Pi)|\Pi \in \mathscr{L}\} \leq T_\lambda(u, v).$$

Recall the definition of $T_\lambda(u, v)$, *i.e.*,

$$T_\lambda(x, y) = \inf\{T_p(\pi) : \pi \text{ is a pass from } x \text{ to } y\}.$$

Let π_0 be the path with minimum delay from u to v, we prove that there exists a cluster to cluster transmission process Π_0, such that $T_p(\Pi_0) = T_p(\pi_0)$.

Assume that $\pi_0 = i_0 i_1 i_2 ... i_K (i_0 = u, i_K = v)$. At time slot t_1, some nodes in path π_0 may be in the same connected cluster as i_0. Let i_{η_1-1} be the node attainable from i_0 with largest subindex, then the link between i_{η_1-1} and i_{η_1} must be off at this time slot. Let $t_2 > t_1$ be the first time slot that this link is on. At time slot t_2, let i_{η_2-1} be the node attainable from i_{η_1} with largest subindex. Then the link between i_{η_2-1} and i_{η_2} must be off until time slot $t_3 > t_2$...At time slot t_k, node $i_{\eta_{k-1}}$ and destination node v are in the same connected cluster and the information transmit to v instantaneously. We denote by Π_0 this cluster to cluster transmission process. And

$$\Pi_0 = \{(t_1, i_0), (t_2, i_{\eta_1}), ..., (t_k, i_{\eta_{k-1}})\}.$$

From the construction of Π_0, it is obvious that $T_p(\Pi_0) = T_p(\pi_0)$.

Thus,

$$\begin{aligned} T_\lambda(u, v) &= T_p(\pi_0) \\ &= T_p(\Pi_0) \\ &\geq \inf\{T_p(\Pi)|\Pi \in \mathscr{L}\}. \quad (13) \end{aligned}$$

Combining Eqn.(24) and Eqn.(13), we obtain

$$T_\lambda(u, v) = \inf\{T_p(\Pi)|\Pi \in \mathscr{L}\}.$$

$\blacksquare$

Lemma 8 establishes the relationship between delay and cluster to cluster transmission process. Cluster to cluster transmission process can represent the information dissemination process more precisely. Our following results on the lower bound of $\gamma(\lambda)$ are just based on the cluster to cluster transmission.

C. Lower Bounds of $\gamma(\lambda)$

In this section, we use the concept of cluster to cluster transmission process to derive a lower bound of $\gamma(\lambda)$.

To start with, we need to introduce a random variable $S_{g,t,u}(\lambda)$(g is the connection function, u is a node) to represent the size of connected cluster in the instantaneous geometric random graph $\mathcal{G}_t(\lambda, r_0, g(r))$. We establish a cartesian coordinate in $\mathbb{R}^2$. We define $H(z_0, a)$ as

$$H(z_0, a) = \{z = (z_x, z_y) \in \mathbb{R}^2| \parallel z - z_0 \parallel < a\}.$$

The random variable $S_{g,t,u}(\lambda)$ is defined as

$$S_{g,t,u}(\lambda) = \sup\{a|\exists \text{ node } v \in H(u, a), v \leftrightarrow u \text{ at time slot } t\}.$$

According to the translation invariance and time independence of our dynamic random connection model $\mathcal{G}_t(\lambda, r_0, g(r))$, $S_{g,t,u}(\lambda)$ is independent of t and u. Thus, we can write $S_{g,t,u}(\lambda)$ as $S_g(\lambda)$ if causing no confusion.
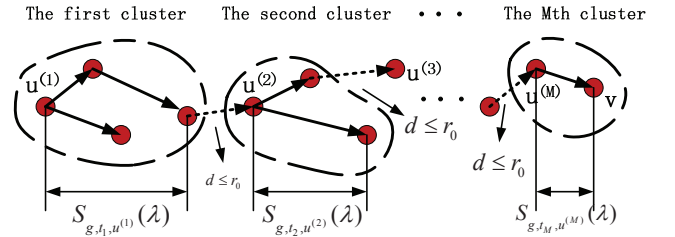

Fig. 3. Illustration of a cluster to cluster transmission process.

Now, given two nodes u(source) and v(destination), consider a cluster to cluster transmission process(Fig. 3)

$$\Pi = \{(u^{(1)}, t_1), (u^{(2)}, t_2), ..., (u^{(M)}, t_M)\},$$

where $u^{(1)} = u$, and $u^{(M)}, v$ are in the same connected cluster at time slot t_M. Then the delay along this cluster to cluster transmission process is

$$T_p(\Pi) = t_M - t_1 = \sum_{k=1}^{M-1}(t_{k+1} - t_k) \geq M - 1.$$

Note that, for $\forall k = 1, 2, ..., M-1$, $u^{(k+1)}$ is connected to a node in w_k, denoted by u'. Then

$$\begin{aligned} \| u^{(k+1)} - u^{(k)} \| &\leq \| u^{(k+1)} - u' \| + \| u' - u^{(k)} \| \\ &\leq S_{g,t_k,u^{(k)}}(\lambda) + r_0. \end{aligned}$$

And for $k = M$,

$$\| v_x - u^{(M)} \| \leq S_{g,t_M,u^{(M)}}(\lambda).$$

Combining the above two inequalities together, we obtain,

$$\begin{aligned} d(u,v) &= \| v - u \| \\ &\leq \sum_{k=1}^{M-1} \| u^{(k+1)} - u^{(k)} \| + \| v - u^{(M)} \| \\ &\leq \sum_{k=1}^{M-1} (S_{g,t_k,u^{(k)}}(\lambda) + r_0) + S_{g,t_M,u_x^{(M)}}(\lambda) \\ &< \sum_{k=1}^{M} (S_{g,t_k,u^{(k)}}(\lambda) + r_0). \end{aligned} \quad (14)$$

For $\forall k$, $S_{g,t_k,u^{(k)}}(\lambda)$ admit the same distribution as $S_g(\lambda)$. Moreover, according to the spatial independence of Poisson Point Process, $S_{g,t_k,u^{(k)}}(\lambda)$ are *i.i.d.* random variables. According to the law of strong large numbers, we have

$$\lim_{M \to \infty} \frac{\sum_{k=1}^{M} (S_{g,t_k,u^{(k)}}(\lambda) + r_0)}{M} = E(S_g(\lambda) + r_0).$$

That is, for $\forall \epsilon > 0$, $\exists M_\epsilon$, such that $\forall M > M_\epsilon$(this condition is satisfied for large enough $d(u,v)$), we have

$$\frac{\sum_{k=1}^{M} (S_{g,t_k,u^{(k)}}(\lambda) + r_0)}{M} < E(S_g(\lambda) + r_0) + \epsilon.$$

Combined with Eqn. (14), we have

$$d(u,v) < M(E(S_g(\lambda) + r_0) + \epsilon).$$

Then

$$T_p(\Pi) \geq M - 1 > \frac{d(u,v)}{E(S_g(\lambda) + r_0) + \epsilon} - 1.$$

Note that the right part of the above equation does not depend on the selection of the cluster to cluster transmission. Thus,

$$T_\lambda(u,v) \geq \frac{d(u,v)}{E(S_g(\lambda) + r_0) + \epsilon} - 1.$$

Therefore,

$$\begin{aligned} \gamma(\lambda) &= \lim_{d(u,v) \to \infty} \frac{T_\lambda(u,v)}{d(u,v)} \\ &\geq \frac{1}{E(S_g(\lambda) + r_0) + \epsilon} \end{aligned} \quad (15)$$

Let $\epsilon \to 0$, we finally obtain

$$\gamma(\lambda) \geq \frac{1}{E(S_g(\lambda) + r_0)}.$$

D. Impact of Transmission Delay

The delay in Large-scale Wireless Network is composed of two parts, *i.e.*, the waiting delay and the propagation delay. In previous sections, we have formulated the waiting delay, while ignore the transmission delay. However, propagation delay may become dominant in some cases especially when the node density is large enough[5]. In the following discussion, we denote by τ the propagation delay for a existing link. For ease of analysis, we assume that the propagation delays are the same for different links. Moreover, we assume that $\tau < 1$. This assumption is reasonable, because if the propagation process along a link cannot be finished in one time slot, the state of the link may change, causing undesirable troubles. Besides, τ is dominated by the bit rate and the length of a message in Large-scale Wireless Networks. Therefore, we can always achieve $\tau < 1$ by slicing messages into small pieces.

After introducing transmission delay, $\gamma(\lambda)$ increases obviously. In this section, we present the proof of theorem 3 as follows.

Proof: We first consider the upper bound. We have already obtained Eqn.(9),*i.e.*,

$$\gamma(\lambda) \leq \kappa \sqrt{\frac{\lambda}{\lambda_L}} E[T_c(e)],$$

where the length of link e is smaller than $\tilde{r} = r_0 \sqrt{\frac{\lambda_L}{\lambda}}(1 + \epsilon)$.

Using similar method in deriving Eqn. (10), we obtain

$$\begin{aligned} E[T_c(e)] &\leq E[T_c(e')] \\ &= \sum_{k=0}^{\infty} (k + \tau)\mathbb{P}(T_c(e') = k) \\ &< \sum_{k=0}^{\infty} (k + 1)(1 - g(\tilde{r}))^k g(\tilde{r}) \\ &= \frac{1}{g(\tilde{r})} \\ &= \frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda}}(1 + \epsilon)\right)} \end{aligned} \quad (16)$$

where e' is link whose length is $\tilde{r}$. The inequality above is slightly different from Eqn. (10). This is because we have taken propagation delay into consideration.

Thus,

$$\gamma(\lambda) \leq \kappa \sqrt{\frac{\lambda}{\lambda_L}} \frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda}}(1 + \epsilon)\right)}.$$

Let $\epsilon \to 0$, we obtain,

$$\gamma(\lambda) \leq \kappa \sqrt{\frac{\lambda}{\lambda_L}} \frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda}}\right)}.$$

---

[5]This is because the waiting delay is caused by the lack of instantaneous connectivity of wireless network. When node density is large enough, the wireless network has instantaneous connectivity, making waiting delay negligible.

Note that $\gamma(\lambda)$ is a monotone decreasing function, thus

$$\gamma(\lambda) \leq \inf_{\lambda' \in [\lambda_L, \lambda]} \kappa \sqrt{\frac{\lambda'}{\lambda_L}} \frac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)}. \quad (17)$$

Then we consider the lower bound. Similar to the previous part, we still focus on the cluster to cluster transmission. Consider a cluster to cluster transmission process

$$\Pi = \{(u^{(1)}, t_1), (u^{(2)}, t_2), ..., (u^{(M)}, t_M)\},$$

similarly, we have, for $\forall k = 1, 2, ..., M-1$,

$$\| u^{(k+1)} - u^{(k)} \| \leq S_{g, t_k, u^{(k)}}(\lambda) + r_0.$$

And for $k = M$,

$$\| v - u^{(M)} \| \leq S_{g, t_M, u_x^{(M)}}(\lambda) < S_{g, t_M, u^{(M)}}(\lambda) + r_0.$$

Besides, the distance transmitted is also limited by the finite hops in one time slot. Since each hop takes τ time slot, then the message can experience at most $\frac{1}{\tau}$ hops in one time slot. As a result, the longest distance transmitted in one time slot is upper bounded by $\frac{r_0}{\tau}$. Then for $\forall k = 1, 2, ..., M-1$,

$$\| u^{(k+1)} - u^{(k)} \| \leq \frac{r_0}{\tau} + r_0.$$

And for $k = M$,

$$\| v - u^{(M)} \| \leq \frac{r_0}{\tau} < \frac{r_0}{\tau} + r_0.$$

Integrating the above four inequalities, we obtain, for $\forall k = 1, 2, ..., M-1$,

$$\| u^{(k+1)} - u^{(k)} \| \leq \min\{S_{g, t_k, u^{(k)}}(\lambda), \frac{r_0}{\tau}\} + r_0.$$

And for $k = M$,

$$\| v - u^{(M)} \| \leq \min\{S_{g, t_M, u^{(M)}}(\lambda), \frac{r_0}{\tau}\} + r_0.$$

Again, using the method in section IV-C, we immediately obtain

$$\gamma(\lambda) \geq \frac{1}{E(\min\{S_g(\lambda), \frac{r_0}{\tau}\}) + r_0}.$$

■

V. DISCUSSION

In this section, we make simulations to uphold our theoretical results. First, we give a further discussion on some parameters in our expressions. Then, enormous simulations are done to justify several assertions in this paper. Our theoretical results are based on a relatively general model, Random Connection Model. Many Network Models can be converted to a Random Connection Model, making our results applicable to many different cases. The difference is that the connection functions are different in different cases. In the following discussion, we simply let $r_0 = 1$, and the connection function g be defined as

$$g(r) = \begin{cases} \frac{1}{4}(2 - r)^2 & : \quad r \leq 1 \\ 0 & : \quad r > 1 \end{cases}$$

Moreover, if we take propagation delay into consideration, we let $\tau = 0.2$.

A. Discussion on Several Parameters

In our expression of theoretical bounds Eqn. (6), two terms κ and $E(S_g(\lambda))$ are applied. Besides, In Eqn. (7), $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$ is applied. κ is a constant, and we simply obtain its value through simulation; while $E(S_g(\lambda))$ and $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$ are functions depend on λ, and we find two analytical expressions to approximate them.

We first focus on κ(defined in lemma 6). In our simulation, we simulated 2304 points in a 40×40 region. The node density is $\lambda = \lambda_L \approx 1.44$, and the transmission range is 1.01. A message is originated from a node located at the center of the region, we record down the minimum number of hops and the distance from the source for each node, and present it in Fig. 4.(a).
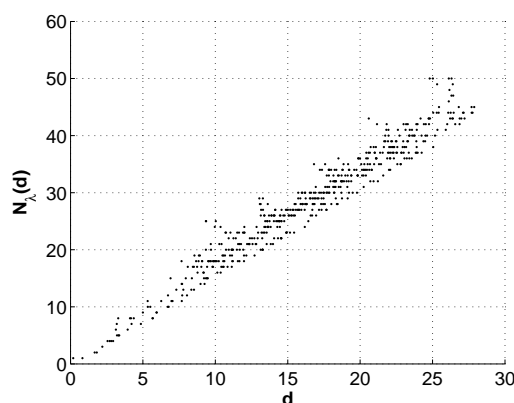
(a) Illustration of the relationship between $N_\lambda(d)$ and d
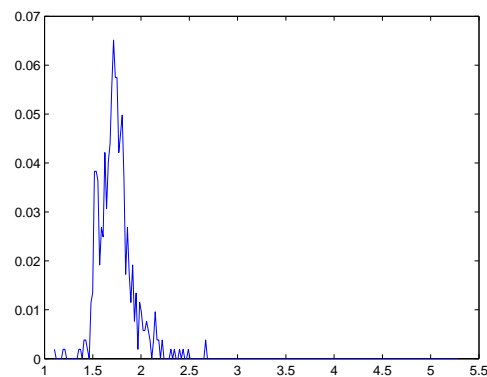
(b) The probability distribution graph of $\frac{N_\lambda(d)}{d}$

Fig. 4. Simulation results on κ. The first Figure reveals the linear relationship between $N_\lambda(d)$ and d, and the second indicates $\kappa \approx 1.7153$.

From Fig. 4.(a), we can see that $N_\lambda(d)$ grows linearly with d. To find κ, we calculate $\frac{N_\lambda(d)}{d}$ for each node, and present its probability distribution graph in Fig. 4.(b). It can be seen from Fig. 4.(b) that the probability $\frac{N_\lambda(d)}{d} = 1.7153$ is the largest. Thus, $\kappa \approx 1.7153$.

Next, we turn to $E(S_g(\lambda))$. The physical meaning of $E(S_g(\lambda))$ is the average size of the connected component

intersected with the origin. It is obvious to note that when $\lambda = 0$, $E(S_g(\lambda)) = 0$. And when $\lambda = \lambda_I$, $E(S_g(\lambda)) = \infty$ since the network is percolated in this case. Thus, we give a conjecture about the analytical expression of $E(S_g(\lambda))$, i.e.,

$$E(S_g(\lambda)) = \frac{c_1 \lambda}{\lambda_I - \lambda}. \tag{18}$$

We make enormous numerical computations to find the experimental values of $E(S_g(\lambda))$ with respect to different λ, ranging from $1.44(\lambda_L \simeq 1.44)$ to 2.4. We then rewrite Eqn. (18) as

$$\frac{1}{E(S_g(\lambda))} = \frac{\lambda_I}{c_1} \cdot \frac{1}{\lambda} - \frac{1}{c_1}.$$

Using least square method, we can easily obtain $c_1 \approx 1.2841$, $\lambda_I \approx 2.4886$.

Then we make a comparison between the fitting value and the experimental value of $E(S_g(\lambda))$. From Fig. 5, It can be seen that there is a good agreement between fitting and experimental results.

Fig. 6. Comparison between experimental and fitting value of $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$.

B. Comparison between Two Bounds

This paper is originated from the idea that the delay-distance ratio $\gamma(\lambda)$ may depend on the node density λ. We calculate $\gamma(\lambda)$ under different node densities. From Fig. 7, we can see that $\gamma(1.6) = \lim_{d \to \infty} \frac{T_\lambda(d)}{d} \approx 0.68$, $\gamma(1.9) \approx 0.35$, $\gamma(2.2) \approx 0.11$, $\gamma(2.5) \approx 0$. This justify the fact that $\gamma(\lambda)$ depend on λ[6], making our discussion meaningful.
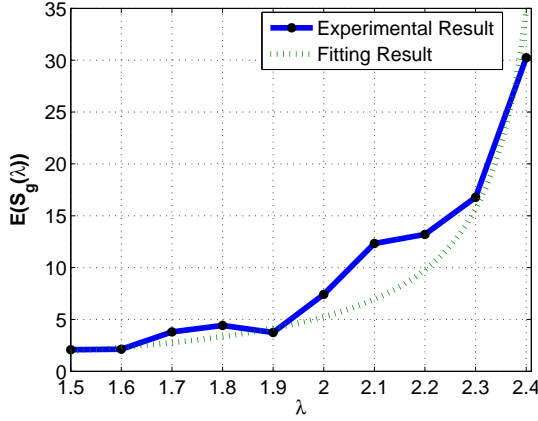

Fig. 5. Comparison between experimental and fitting value of $E(S_g(\lambda))$.

Now, we come to $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$. Obviously, $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\}) = 0$ whenever $\lambda = 0$. Besides, $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\}) \le \frac{r_0}{\tau}$ for $\forall \lambda$, and $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$ is monotone increasing with λ. Thus, we conjecture that the analytical expression of $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$ has the format

$$E(\min\{S_g(\lambda), \frac{r_0}{\tau}\}) = \frac{c_2 \lambda}{c_3 + \lambda}. \tag{19}$$

Similarly, we make enormous numerical computations to find the experimental values of $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$ with respect to different λ, ranging from 1.44 to 20. We rewrite Eqn. (19) as

$$\frac{1}{E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})} = \frac{c_3}{c_2} \cdot \frac{1}{\lambda} + \frac{1}{c_2}.$$

Using least square method, we can easily obtain $c_2 \approx 2.0845$, $c_3 \approx 1.6813$.

Then we make a comparison between the fitting value and the experimental value of $E(\min\{S_g(\lambda), \frac{r_0}{\tau}\})$. From Fig. 6, It can be seen that there is a good agreement between fitting and experimental results.
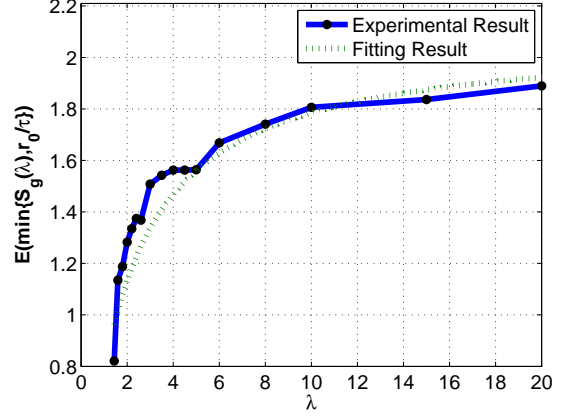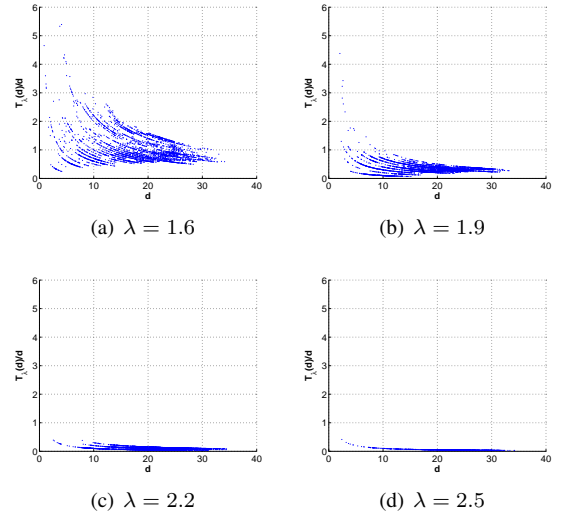

Fig. 7. Simulation results on different λ.

Now, we are ready to compare our theoretical bounds and the experimental values of $\gamma(\lambda)$. We first ignore the propagation delay, Fig. 8 shows the comparison between the experimental value and our theoretical value of both the upper bound and the lower bound. In our simulations, We work out the experimental values of $\gamma(\lambda)$ where λ is set to be evenly distributed in the interval of $[\lambda_L, \lambda_I]$. From Fig. 8, we find the

[6]To be more exact, $\gamma(\lambda)$ decreases with λ.

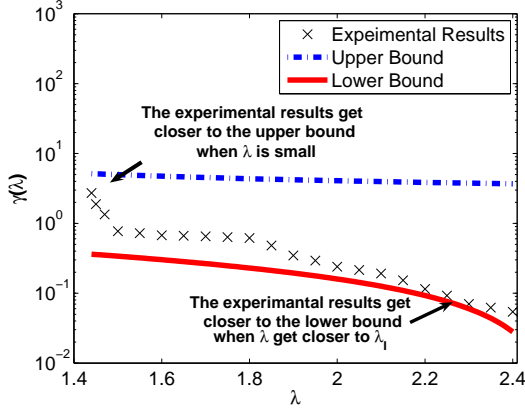experimental values are right bounded by both the upper and the lower bounds.

Fig. 8. Comparison between upper bound and lower bound(propagation delay is ignored).

Then, we take propagation delay into consideration. In the following part, we take $\tau = 0.2$. We first examine the effect of introducing propagation delay to the delay-distance ratio $\gamma(\lambda)$.

We consider two networks, whose node densities λ are 1.5 and 2.2(1.5 is near λ_L, 2.2 is near λ_I) respectively. For each network, we make simulations to find $\gamma(\lambda)$ in the cases $\tau = 0$ and $\tau = 0.2$. The result is show in Fig. 9.
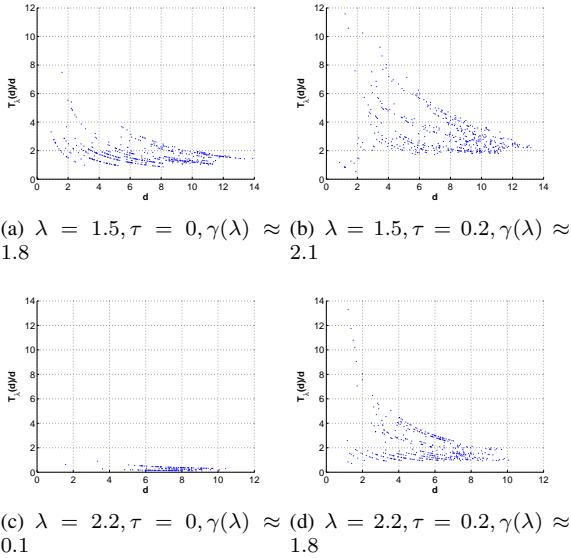

(a) $\lambda = 1.5, \tau = 0, \gamma(\lambda) \approx 1.8$ (b) $\lambda = 1.5, \tau = 0.2, \gamma(\lambda) \approx 2.1$

(c) $\lambda = 2.2, \tau = 0, \gamma(\lambda) \approx 0.1$ (d) $\lambda = 2.2, \tau = 0.2, \gamma(\lambda) \approx 1.8$

Fig. 9. The influence of propagation delay τ to the delay-distance ratio $\gamma(\lambda)$.

From Fig. 9, we can also see that when λ is small, the influence of τ to $\gamma(\lambda)$ is small; and the influence becomes more significant as λ grows larger. This also indicate that when node density is small, delay is mainly caused by the waiting delay(waiting delay is mainly caused by the lack of

instantaneous connectivity); while when node density is large, delay is mainly caused by propagation delay.

Next, we compare our theoretical bounds and experimental results of $\gamma(\lambda)$ in the case $\tau = 0.2$. In our simulation, the node densities λ are chosen from $[1.44, 20]$. As the change of $\gamma(\lambda)$ is larger when λ is small, we choose more simulation points for smaller node density. The comparison between our theoretical bounds and experimental results is shown in Fig. 10. The experimental values are right bounded by both the upper and the lower bounds.
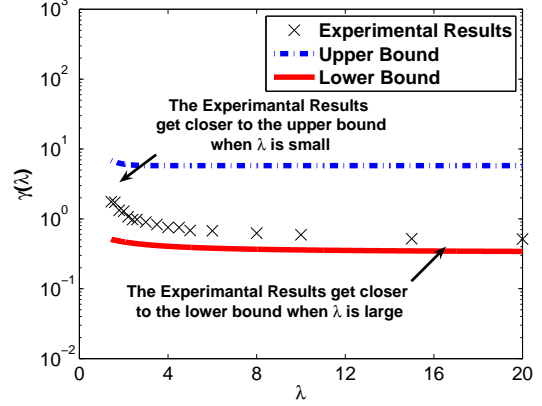

Fig. 10. Comparison between upper bound and lower bound(propagation delay is considered).

From Fig. 8 and Fig. 10, it can be also seen that when λ is large, $\gamma(\lambda)$ is much closer to the lower bound. An explanation to this phenomena is that the larger $\gamma(\lambda)$ is, the larger the size of clusters in the cluster to cluster transmission process. Larger cluster size provides more opportunity to forward messages. Thus, when the node density is large enough, it is probably that the message can transmit again right at the next time slot. This makes our lower bound more accurate. However, our upper bound is a little bit loose. But we can see that $\gamma(\lambda)$ gets closer to the upper bound when the node density λ decreases. This is because our upper bound is obtained from the delay of just one path. And the smaller λ is, the smaller the number of paths connecting two nodes. This makes $\gamma(\lambda)$ get closer to the upper bound.

VI. CONCLUSION

In this paper, we study the tradeoff between $\gamma(\lambda)$ and λ using percolation theory. We point out that the lack of instantaneous connectivity brings about waiting delay and prove that $\gamma(\lambda)$ is upper bounded by $\inf\limits_{\lambda' \in [\lambda_L, \lambda]} \kappa \sqrt{\dfrac{\lambda'}{\lambda_L}} \left(\dfrac{1}{g\left(r_0 \sqrt{\frac{\lambda_L}{\lambda'}}\right)} - 1 \right)$, and lower bounded by $\dfrac{1}{E(S_g(\lambda) + r_0)} \leq \gamma(\lambda)$. Then we take propagation delay into consideration, and obtain further results. Finally, through simulations based on the exact value of $\gamma(\lambda)$, we further obtain a new observation that the lower bound serves as a good estimate to the value of $\gamma(\lambda)$ in dense

networks. And $\gamma(\lambda)$ gets closer to the upper bound when λ decreases. Simulation results conform our theoretical findings.

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APPENDIX

Appendix I

The proof of Lemma 6 is presented as follows. The method used in this proof is similar to that used by Dousse *et al.* in [6] and Kong *et al.* in [7].

We first construct a cartesian coordinate system. Without of loss of generality, we assume that there is a node at the

origin. Let $z_n = \arg\min_{z \in \mathcal{C}(\mathcal{B}(\lambda_L,(1+\epsilon)r_0))}\{d(z,(0,n))\}$, then we have the following lemma.

Lemma 9. $d(z,(0,n)) < \infty$ *with probability 1.*

We will present the proof of lemma 9 in Appendix II. Let $N'_{\lambda_L}(m,n) = N_{\lambda_L}(d(z_n,z_m))$We need first to prove that $N'_{\lambda_L}(0,m)$ scales linearly with respect to m, *i.e.,*

Lemma 10. *There exists* κ*, such that*

$$\lim_{m \to \infty} \frac{N'_{\lambda_L}(0,m)}{m} = \kappa.$$

To prove lemma 10, we use lemma 7.

It is easy to see that $N'_{\lambda_L}(0,m) \leq N'_{\lambda_L}(0,l) + N'_{\lambda_L}(l,m)(0 \leq l \leq m)$. Then the first condition of lemma 7 is satisfied. Since Poisson Boolean Model $\mathcal{B}(\lambda_L,(1+\epsilon)r_0))$ is homogeneous, the second and the third conditions of lemma 7 are also satisfied. Now, we only need to prove that conditions 4 and 5 are also satisfied.

Lemma 11. $E(N'_{\lambda_L}(0,m)) < \infty$.

Proof: Consider $N'_{\lambda_L}(0,m)$, let $z' = \frac{z_0+z_m}{2}$. We draw a series of squares centering at z'(Fig. 11), and the side lengths are $1, 2, 4, ..., 2^k, ...$
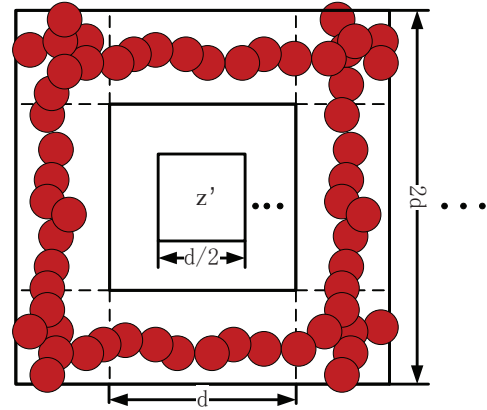

Fig. 11. A series of squares centering at z'.

We use $R(d)$ to denote the rectangle with side lengths $\frac{d}{2}$ and $2d$. We say $R(d)$ is good if and only if there exists a crossing connecting the two short sides in $R(d)$(Fig. 12). We denote the event that $R(d)$ is good by $A_R(d)$. Since the Poisson Boolean Model $\mathcal{B}(\lambda_L,(1+\epsilon)r_0)$ is percolated[7],

$$\lim_{d \to \infty} \mathbb{P}(A_R(d)) = 1. \tag{20}$$

We use $C(d)$ to denote the square torus $([z'_x - d, z'_x + d] \times [z'_y - d, z'_y + d]) \setminus ([z'_x - \frac{d}{2}, z'_x + \frac{d}{2}] \times [z'_y - \frac{d}{2}, z'_y + \frac{d}{2}])$. We say $C(d)$ is good if and only if there exists a circuit in $C(d)$(Fig.

---

[7]λ_L is the critical density of Poisson Boolean Models with transmission range r_0. Now the transmission range is $(1+\epsilon)r_0$, according to lemma 5, we know the critical density becomes $(1+\epsilon)^{-2}\lambda_L$. Obviously, $\lambda_L > (1+\epsilon)^{-2}\lambda_L$, thus $\mathcal{B}(\lambda_L,(1+\epsilon)r_0)$ is percolated.
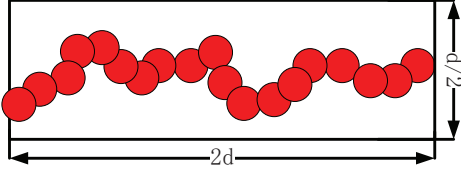
Fig. 12. Illustration of a good rectangle $R(d)$.

13). We denote the event that $C(d)$ is good by $A_C(d)$. From Fig. 13, we can see that $C(d)$ is composed of four $\frac{d}{2} \times 2d$ rectangles, i.e., $R_1(d) = [z'_x - d, z'_x - \frac{d}{2}] \times [z'_y - d, z'_y + d]$, $R_2(d) = [z'_x + \frac{d}{2}, z'_x + d] \times [z'_y - d, z'_y + d]$, $R_3(d) = [z'_x - d, z'_x + d] \times [z'_y + \frac{d}{2}, z'_y + d]$, $R_4(d) = [z'_x - d, z'_x + d] \times [z'_y - d, z'_y - \frac{d}{2}]$. And we use $A_{R_i}(d)(i = 1, 2, 3, 4)$ to represent the event that $R_i(d)$ is good.
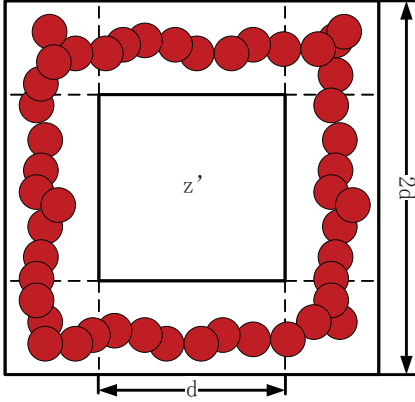

Fig. 13. Illustration of a good square torus $C(d)$. Here, $z' = (z'_x, z'_y)$.

Obviously, if for $\forall i = 1, 2, 3, 4$, $R_i(d)$ is good, $C(d)$ must be good. Thus,

$$\mathbb{P}(A_C(d)) \geq \mathbb{P}(A_{R_1}(d) \bigcap A_{R_2}(d) \bigcap A_{R_3}(d) \bigcap A_{R_4}(d)).$$

Note that for $\forall i = 1, 2, 3, 4$, $A_{R_i}(d)$ is an increasing event[8]. According to the FKG Inequality(Theorem 2.4 in [5]), we have

$$
\begin{aligned}
\mathbb{P}(A_C(d)) &\geq \mathbb{P}(A_{R_1}(d) \bigcap A_{R_2}(d) \bigcap A_{R_3}(d) \bigcap A_{R_4}(d)) \\
&\geq \mathbb{P}(A_{R_1}(d))\mathbb{P}(A_{R_2}(d))\mathbb{P}(A_{R_3}(d))\mathbb{P}(A_{R_4}(d)) \\
&= \mathbb{P}(A_R(d))^4. \quad (21)
\end{aligned}
$$

Combine Eqn. (20) with Eqn. (21), and we obtain

$$\lim_{d \to \infty} \mathbb{P}(A_C(d)) = 1.$$

Thus, for $\forall \frac{1}{2} < \rho < 1$, there exists d_ρ, such that, for $\forall d \geq d_0$,

$$\mathbb{P}(A_C(d)) \geq \rho.$$

[8]A random variable N is on the measurable pair $(\Omega, \mathscr{F})$ is called increasing if $N(\omega) \leq N(\omega')$ whenever $\omega \leq \omega'$.

Let $k_0 = \min\{k | 2^k \geq \| z_m - z_0 \|, 2^k \geq d_\rho\}$, then for all $k \geq k_0$,

$$\mathbb{P}(A_C(2^k)) \geq \rho.$$

Assume that $C(2^k)(k \geq k_0)$ is good, if the shortest path from z_0 to z_m is not contained in the square $[z'_x - 2^k, z'_x + 2^k] \times [z'_y - 2^k, z'_y + 2^k]$, it must intersect with the circuit in $C(2^k)$(Fig 14). We can replace the part of the path ACB with ADB, then the resulting path is shorter. Thus, the shortest path from z_0 to z_m must be contained in $[z'_x - 2^k, z'_x + 2^k] \times [z'_y - 2^k, z'_y + 2^k]$.
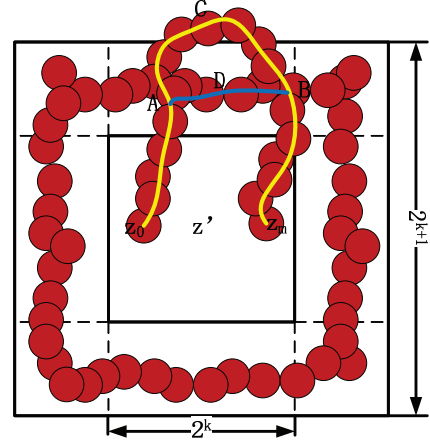

Fig. 14. The contradiction if the shortest path from z_0 to z_m is not contained in $[z'_x - 2^k, z'_x + 2^k] \times [z'_y - 2^k, z'_y + 2^k]$.

Suppose u, v, w are three consecutive nodes along this shortest path. Then $\| u - w \| > (1 + \epsilon)r_0$, or we can eliminate node v, and get a shorter path. This also indicates that if we draw disks with radius $\frac{(1+\epsilon)r_0}{2}$ centering at u and w respectively, the two disks are disjoint. Assume the number of hops of the shortest path is N, then we can draw $\frac{L}{2}$ disjoint disks in total. And these disks are all located in a square with side length $2^{k+1} + (1 + \epsilon)r_0$. Thus,

$$\frac{L}{2} \cdot \pi(\frac{(1+\epsilon)r_0}{2})^2 \leq (2^{k+1} + (1+\epsilon)r_0)^2.$$

Then,

$$L \leq \frac{8(2^{k+1} + (1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2}.$$

Note that $N'_{\lambda_L}(0, m)$ is the minimum number of hops from x_0 to x_m, thus

$$N'_{\lambda_L}(0, m) \leq L \leq \frac{8(2^{k+1} + (1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2}.$$

Now, if $N'_{\lambda_L}(0, m) > \frac{8(2^{k+1} + (1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2}$, then none of $C(2^{k_0+1}), C(2^{k_0+2}), ..., C(2^k)$ is good. Thus,

$$\mathbb{P}(N'_{\lambda_L}(0, m) > \frac{8(2^{k+1} + (1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2})$$

$$\leq \prod_{i=k_0+1}^{k} \mathbb{P}(A_C^c(2^i)) \leq (1 - \rho)^{k-k_0}.$$

Let $l_k = \frac{8(2^{k+1}+(1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2}$, then,

$$E(N_{\lambda_L}^{'}(0,m))$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(N_{\lambda_L}^{'}(0,m) \geq i)$$

$$= \sum_{i=1}^{l_{k_0}} \mathbb{P}(N_{\lambda_L}^{'}(0,m) \geq i) + \sum_{j=k_0}^{\infty} \sum_{i=l_j+1}^{l_{j+1}} \mathbb{P}(N_{\lambda_L}^{'}(0,m) \geq i)$$

$$\leq l_{k_0} + \sum_{j=k_0}^{\infty} \mathbb{P}(N_{\lambda_L}^{'}(0,m) > l_j) \cdot (l_{j+1} - l_j)$$

$$< l_{k_0} + \sum_{j=k_0}^{\infty} (1-\rho)^{j-k_0} \cdot \frac{8(2^{j+2}+(1+\epsilon)r_0)^2}{\pi((1+\epsilon)r_0)^2}$$

$$< \infty \qquad .$$

∎

Then condition 4 of lemma 7 is satisfied. Next, we prove that $N_{\lambda_L}^{'}(m,n)$ satisfies condition 5. To demonstrate that $N_{\lambda_L}^{'}(mk,(m+1)k)$ is ergodic, we show it is strong mixing, which is a stronger property.

Lemma 12. $N_{\lambda_L}^{'}(mk,(m+1)k)$ *is strong mixing.*

Proof: In previous analysis, we have proved that $\mathbb{P}(A_C(2^k)) \geq \rho$ whenever $k \geq k_0$. Summing over k yields

$$\sum_{k=k_0}^{\infty} \mathbb{P}(A_C(2^k)) = \sum_{k=k_0}^{\infty} \rho = \infty. \qquad (22)$$

Since $A_C(2^k), k = k_0, k_0+1, ...$ are independent events, according to the Borel-Cantelli Theorem, with probability 1, there exist $k^{'} < \infty$[9], such that $A_C(2^{k^{'}})$ occurs.

We now construct squares B_1 and B_2 centered at $\frac{x_{mk}+x_{(m+1)k}}{2}$ and $\frac{x_{(m+n)k}+x_{(m+n+1)k}}{2}$ with side length $2^{k^{'}+1}$ and $2^{k^{''}+1}$ respectively, such that the path with minimum number of hops from x_{mk} to $x_{(m+1)k}$, and the path with minimum number of hops from $x_{(m+n)k}$ to $x_{(m+n+1)k}$ are contained in B_1 and B_2.

Due to the stationarity, $k^{''}$ does not rely on n. Besides, $k^{'}$ and $k^{''}$ are all finite(we denote this event by A_f) with probability 1. Thus, when n is large enough, B_1 and B_2 are disjointed. Hence, $N_{\lambda_L}^{'}(mk,(m+1)k)$ and $N_{\lambda_L}^{'}((m+n)k,(m+n+1)k)$ become independent.

Therefore,

$$\lim_{n\to\infty} \mathbb{P}(\{N_{\lambda_L}^{'}(mk,(m+1)k) < i\} \bigcap \{N_{\lambda_L}^{'}((m+n)k,(m+n+1)k) < i\})$$

$$= \lim_{n\to\infty} \mathbb{P}(\{N_{\lambda_L}^{'}(mk,(m+1)k) < i\} \bigcap \{N_{\lambda_L}^{'}((m+n)k,(m+n+1)k) < i\}|A_f)\mathbb{P}(A_f)$$

$$+ \lim_{n\to\infty} \mathbb{P}(\{N_{\lambda_L}^{'}(mk,(m+1)k) < i\} \bigcap \{N_{\lambda_L}^{'}((m+n)k,(m+n+1)k) < i\}|A_f^c)\mathbb{P}(A_f^c)$$

[9]Actually, the number of such $k^{'}$ is infinite.

$$= \mathbb{P}(\{N_{\lambda_L}^{'}(mk,(m+1)k) < i\}|A_f)\,\mathbb{P}(\{N_{\lambda_L}^{'}((m+n)k,(m+n+1)k) < i\}|A_f)$$

$$= \mathbb{P}(\{N_{\lambda_L}^{'}(mk,(m+1)k) < i\})\,\mathbb{P}(\{N_{\lambda_L}^{'}((m+n)k,(m+n+1)k) < i\}) \qquad .$$

Then, $N_{\lambda_L}^{'}(mk,(m+1)k)$ is strong mixing. ∎

Now, we have proved that $N_{\lambda_L}^{'}(m,n)$ satisfy conditions $1-5$ of lemma 7. Thus, there exists κ, such that

$$\lim_{m\to\infty} \frac{N_{\lambda_L}^{'}(0,m)}{m} = \kappa.$$

Now, we are ready to prove lemma 6.

Proof: Consider $N_{\lambda_L}(d(u,v))$. Without of loss of generality, we suppose that u is at the origin, v is at the $+x$ axis. Assume that integer n_d satisfy $n_d \leq d(u,v) < n_d + 1$, then

$$N_{\lambda_L}(d(u,v)) \leq N_{\lambda_L}(d(0,z_{n_d})) + N_{\lambda_L}(d(z_{n_d},v)), \quad (23)$$

and

$$N_{\lambda_L}(d(u,v)) \geq N_{\lambda_L}(d(0,z_{n_d})) - N_{\lambda_L}(d(z_{n_d},v)). \quad (24)$$

Note that

$$\begin{aligned} d(z_{n_d},v) &\leq d(z_{n_d},(0,n_d)) + d((0,n_d),v) \\ &\leq d(z_{n_d},(0,n_d)) + 1 \\ &< \infty. \end{aligned}$$

Using similar method in the proof of lemma 11, we can prove $E(N_{\lambda_L}(d(z_{n_d},v))) < \infty$(to avoid verbosity, we do not elaborate it here). Therefore, $N_{\lambda_L}(d(z_{n_d},v)) < \infty$ with probability 1. Then, divide Eqn. (23) and Eqn. (24) by $d(u,v)$, and let $d(u,v) \to \infty$. We immediately obtain

$$\lim_{d(u,v)\to\infty} \frac{N_{\lambda_L}(d(u,v))}{d(u,v)} = \kappa.$$

∎

Appendix II

We present the proof of lemma 9 here.

Proof: Similar to that in the proof of lemma 11, we construct a series of squares centered at $(0,n)$ with side length $1, 2, 4, ..., 2^k, ...$(Fig. 11). $C(d)$ is defined similarly. We say $C(d)$ is perfect if and only if $C(d)$ is good and the circuit in $C(d)$ belongs to the giant component.

Again, applying Borel-Cantelli Theorem to Eqn. (22), there exists a sequence $\{k_i^{'}\}$, such that for $\forall i = 1, 2, ...$, event $A_C(2^{k_i^{'}})$ occurs.

Suppose that the probability that $d(z_n,(0,n)) < \infty$ is smaller than 1. Then, in the case $d(z_n,(0,n)) = \infty$, none of $C(2^{k_i^{'}})(i = 1, 2, ...)$ is perfect, otherwise $d(z_n,(0,n))$ is finite.

This also indicates that there exists an infinite large circuit encircling $(0,n)$, and this circuit does not belong to the giant component. Therefore, with a probability larger than 0, there exist two infinite connected cluster in $\mathcal{B}(\lambda_L, (1+\epsilon)r_0)$. This contradict the uniqueness of infinite connected cluster in percolated networks.

Thus, $d(z_n,(0,n)) < \infty$ with probability 1. ∎